Monetary Economics

Dynamic Programming in a Stochastic Environment

Gerald P. Dwyer

September 2021

Outline

Stochastic Infinite-Horizon Problem

- World is not really perfectly predictable
- Allow for risk of alternative outcomes

Stochastic Aspect of Economy

- Introduce probability of two different production functions each period
  - $y_t = A_1 f(k_t)$ with probability p_1
  - $y_t = A_2 f(k_t)$ with probability p_2
  - where $A_1 > A_2$ is a technological parameter
  - State 1 is the good state and state 2 is the bad state
  - Assume that the state in any period is independent of the state in any other period
  - A is often called a technology shock
- This technology shock is a "stochastic process" because it evolves over time according to a probability law
- The household determines consumption this period after the value of $A(t)$ is revealed

Budget Constraint with Stochastic Technology

- With the technology shock $A(t)$ the budget constraint now is

$$k_{t+1} = A(t) f(k_t) + k_t - c_t$$

  - where $A(t) = A_1$ with probability p_1 or $A(t) = A_2$ with probability p_2

Implications of Random Aspect of Economy

- This seemingly small change has dramatic implications for the maximization problem
- At the start of period t, the value of $A(t)$ is not known
  - The value of A in t becomes known and the values of consumption this period and the capital stock next period will be different depending on whether A is high or low
  - $A(t) = A_1$ or $A(t) = A_2$
- At the start of period $t+1$, the capital stock $k(t+1)$ will reflect whether the value of $A(t)$ was high or low
  - The chosen values of c_{t+1} and k_{t+2} will depend on $A(t)$ in the past
  - They also will depend on $A(t+1)$, just like c_t and k_{t+1} depend on $A(t)$
- At the start of period $t+2$, the capital stock k_{t+2} will reflect that the values of $A(t)$ and $A(t+1)$ were high or low in those two periods
  - The chosen values of c_{t+2} and k_{t+3} will depend on $A(t)$, $A(t+1)$ and $A(t+2)$
- Looks incredibly complicated

Repetition in Economy

- Once one knows the capital stock at the start of a period, each period looks the same
  - Utility function is the same

$$u\left(c_t\right) \text{ is the same function for all } t$$

- Budget constraint is the same

$$c_t + k_{t+1} = A\left(t\right) f\left(k_t\right) + k_t$$

- Everything one wants to know about past values of A at the start of period t is summarized in k_t
  - This is true for all periods

Maximization Problem

- At period t with perfect foresight, maximize

$$U_t = \sum_{i=0}^{\infty} \beta^i u\left(c_{t+i}\right)$$

  - which could be written

$$U_t = \sum_{i=0}^{\infty} \beta^i u\left[f\left(k_{t+i}\right) - k_{t+1+i} + k_{t+i}\right]$$

  - Want to maximize this with respect to k_{t+1+i} for all $t+i$
- Don't know even c_t and k_{t+1} for sure, let alone future values
- Maximize

$$\mathrm{E}_t\, U_t = \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i u\left(c_{t+i}\right)$$

  - where E_t indicates the expected value conditional on information available at the start of period t
  - This information available includes the value of $A\left(0\right)$

Maximization of Expected Utility

- Can maximize expected utility in the usual way subject to the budget constraint
- Maximize

$$\mathrm{E}_t\, U_t = \mathrm{E}_t \sum_{i=0}^{\infty} \beta^i u\left(c_{t+i}\right)$$

- Again the value function is helpful
- Will start at $t = 0$

Value Function given $A(0) = A_1$

- Define the value function, the maximum expected utility at $t = 0$ given k_0 and $A(t) = A_1$

$$V(k_0, A_1) = \max_{\{c_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$k_1 = A_1 f(k_0) + k_0 - c_0$$

and the budget constraint for future periods

$$k_{t+1} = A(t) f(k_t) + k_t - c_t$$

- Define a similar function for $A(0) = A_2$

Value Function given $A(0) = A_2$

- Define the value function, the maximum expected utility given k_0 and $A(0) = A_2$

$$V(k_0, A_2) = \max_{\{c_t\}_{t=0}^{\infty}} \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$k_1 = A_2 f(k_0) + k_0 - c_0$$

and the budget constraint for future periods

$$k_{t+1} = A(t) f(k_t) + k_t - c_t$$

Value Function

- Expected utility is a function of two state variables, k and A
  - Note that k_0 is predetermined in period 0 and A is not controllable by the agent
- The value function as

$$V(k_0, A(0)) = \max_{c_0} \left[u(c_0) + \beta \, \mathrm{E}_0 \, V(k_1, A(1)) \right]$$

- The problem is recursive with the problem each period looking the same given the solution for the prior period
- The expectation operator is not applied to the maximization of consumption this period because the solution is conditional on the realization of the state variable $A(0)$
- The expectations operator appears because for any chosen value of k_1, denoted $\hat{k}_1$,

$$\mathrm{E}_0 \, V\left(\hat{k}_1, A(1)\right) = p_1 V\left(\hat{k}_1, A_1\right) + p_2 V\left(\hat{k}_1, A_2\right)$$

Value Function For Any Period t

- Can write the problem for any period as

$$V\left(k_t, A\left(t\right)\right) = \max_{c_t}\left[u\left(c_t\right) + \beta \, \mathrm{E}_t \, V\left(k_{t+1}, A\left(t+1\right)\right)\right]$$

  - subject to

$$k_{t+1} = A\left(t\right) f\left(k_t\right) + k_t - c_t$$

- The solution is a function that gives the values of the control variables that maximize the value function over the domain of the state variables
- Can write the solution as

$$k_{t+1} = H\left(k_t, A\left(t\right)\right)$$

- Called a "state-contingent plan" because the actual outcomes will be subset of the possible outcomes

Summary

- A common way to introduce a stochastic element is a "technology shock"
  - Could also introduce a stochastic element in the value of assets
- This stochastic aspect of the economy implies that any optimal plan is state contingent
- The optimal plan can be found by maximizing expected utility in the usual way
- The value function reduces the apparent algebraic complexity
- The value function also is an important part of finding numerical solutions
  - Finding numerical solutions uses the fact that the value function is a difference equation in the value function in function space

$$V\left(k_t, A\left(t\right)\right) = \max_{c_t}\left[u\left(c_t\right) + \beta\, \mathrm{E}_t\, V\left(k_{t+1}, A\left(t+1\right)\right)\right]$$

  - subject to the budget constraint