Monetary Economics

General Equilibrium with Infinite-Horizon Consumers

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Outline

Reading

- McCallum 1984 JPE

Economy

- Representative agent model
  - Everyone is the same
  - Any differences from the average are irrelevant
- Agents
  - Households
  - Government
- One commodity model
- Population constant

Agents' Activities

- Households maximize utility
  - Consume and hold money
  - Real consumption per person c_t
  - Money holdings
  - Leisure not included in utility function
- Money holdings
  - M_t is the nominal stock of money at the start of the period
  - Number of dollars measured, e.g. 5 trillion dollars
  - P_t is the price level
    - P_t is the price of the commodity
    - Dollars per unit of output
  - M_t / P_t is the real stock of money
- Bond holdings
  - Households hold one-period bonds issued by the government

Maximization of Utility

- The representative household maximizes utility given by

$$U_t = u(c_t, m_t) + \beta u(c_{t+1}, m_{t+1}) + \beta^2 u(c_{t+2}, m_{t+2}) + \dots$$

- For all t

$$\begin{aligned} u_c(c_t, m_t) &> 0 \text{ and } u_m(c_t, m_t) > 0 \\ u_{cc}(c_t, m_t) &< 0 \text{ and } u_{mm}(c_t, m_t) < 0 \\ u_{cm}(c_t, m_t) &= 0 \end{aligned}$$

- Common additional assumptions
  - $\lim_{c\to 0} u_c(c_t, m_t) = \infty$ and $\lim_{c\to\infty} u_c(c_t, m_t) = 0$
  - $\lim_{m\to 0} u_m(c_t, m_t) = \infty$and $\lim_{m\to\infty} u_m(c_t, m_t) = 0$
  - Guarantees positive amounts of both but a finite amount in equilibrium
- May want to impose other things such as
  - Friedman: There is an $\bar{m} > 0$ such that $u_m(c_t, m_t) < 0$ for all $m > \bar{m}$

Utility and Discounting

- β is the discount factor
- $\beta = \frac{1}{1+\delta}$ where δ is the discount rate
  - Similar to an interest rate in a present-value formula

Budget Constraint

- Cannot set this up without knowing more about the economy
- Need to know opportunity set in addition to preferences
  - Income to pay for consumption
  - Government
- Each household has access to a production function that is homogeneous of degree one in labor and capital
- Labor is supplied inelastically, so it is proportional to population N
- Capital can change over time due to investment in new capital
- Production function

$$y_t = f(k_t)$$

  with $f_k(k_t) > 0$ and $f_{kk}(k_t) < 0$
- k_t is used to produce output in period t so this is capital at the start of the period
- Common to assume
  - $\lim_{k \to 0} f_k(k_t) = \infty$ and $\lim_{k \to \infty} f_k(k_t) = 0$
  - Guarantees positive, finite amount in equilibrium

Government

- Money comes from somewhere
- Suppose it is fiat money produced by the government and distributed as lump-sum transfers to households
- Real per capita lump-sum transfers net of taxes is $\nu_t \gtreqless 0$
  - Nominal per capita lump-sum transfers net of taxes is $P_t \times \nu_t$
  - P_t is the price level – price of output, such as dollars per unit of the good that is output
  - No marginal tax rate
- Government also prints money and issues bonds

Government Budget Constraint

- Spending - revenue = receipts from bonds + revenue from creating money
- Spending - revenue = transfers net of taxes
  - No government purchases
  - Transfers net of taxes is "Spending - revenue"
  - Nominal terms $P_t v_t$
  - Real terms v_t

Nominal Revenue from Printing Money

- Nominal revenue from printing money

$$M_{t+1} - M_t$$

  - Money is the stock at the start of the period

Real Revenue from Printing Money

- Real terms
$$\frac{M_{t+1} - M_t}{P_t} = \frac{M_{t+1}}{P_t} - m_t$$
  - Note that
$$\frac{M_{t+1}}{P_t} = \frac{M_{t+1}}{P_{t+1}} \frac{P_{t+1}}{P_t} = m_{t+1} \frac{P_{t+1}}{P_t}$$
  - and
$$\text{Define } \pi_t = \frac{P_{t+1} - P_t}{P_t}$$
$$\pi_t = \frac{P_{t+1}}{P_t} - 1$$
$$1 + \pi_t = \frac{P_{t+1}}{P_t}$$

Real Revenue from Printing Money

- Have

$$\frac{M_{t+1} - M_t}{P_t} = \frac{M_{t+1}}{P_t} - m_t$$
$$\frac{M_{t+1}}{P_t} = m_{t+1} \frac{P_{t+1}}{P_t}$$
$$\frac{P_{t+1}}{P_t} = 1 + \pi_t$$

- which implies the real revenue from printing money is

$$\frac{M_{t+1} - M_t}{P_t} = m_{t+1} (1 + \pi_t) - m_t$$

Bond Issuance

- The government issues bonds and promises to pay the interest payment and the principal payment at expiration
- One-period bonds
  - Could have have a period greater than one but just would introduce a lot of accounting and no economics

Bond Issuance

- The government issues bonds and promises to pay the interest payment and the principal payment at expiration
- Bonds are discount bonds so no explicit interest payment
- Households can buy bonds during period t for the price Q_t
- The bonds are redeemed for one unit of money in period $t+1$
- The nominal holding period return is

$$R_t = \frac{1 - Q_t}{Q_t}$$

- The real rate r_t is given by

$$1 + R_t = (1 + r_t)(1 + \pi_t)$$

  - which is implied by defining the real price in t to be Q_t / P_t and the real payment in $t+1$ to be $1/P_{t+1}$

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  - For you to do: Show that the definitions imply the real return is given by the equation above

Net Receipts from Issuing Bonds

- The government pays off B_t bonds at one unit of money per bond at the start of period t
  - The real outlay is one unit of money times $B_t / P_t = b_t$ in period t for redeeming old bonds
- The government issues B_{t+1} bonds at the price Q_t at the start of period t
  - The real receipts are $Q_t B_{t+1} / P_t$ for issuing new bonds
  - and

$$\frac{Q_t B_{t+1}}{P_t} = \frac{Q_t}{1} \frac{B_{t+1}}{P_{t+1}} \frac{P_{t+1}}{P_t}$$

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  - and

$$\frac{Q_t}{1} \frac{B_{t+1}}{P_{t+1}} \frac{P_{t+1}}{P_t} = \left(\frac{1}{1 + R_t} \right) b_{t+1} \left(1 + \pi_t\right)$$

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  - and since $\frac{1+\pi_t}{1+R_t} = \frac{1}{1+r_t}$,

$$\frac{Q_t B_{t+1}}{P_t} = \left(1 + r_t \right)^{-1} b_{t+1}$$

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$$\frac{Q_t B_{t+1}}{P_t} = (1 + r_t)^{-1} b_{t+1}$$

- Therefore the net receipts from issuing bonds are given by

$$(1 + r_t)^{-1} b_{t+1} - b_t$$

Government Budget Constraint

- The government's budget constraint in real terms is given by

$$v_t = \frac{(M_{t+1} - M_t)}{P_t} + \frac{Q_t B_{t+1}}{P_t} - \frac{B_t}{P_t}$$

- which also is given by

$$v_t = (1 + \pi_t)\, m_{t+1} - m_t + (1 + r_t)^{-1}\, b_{t+1} - b_t$$

Receipts and Resources to Acquire Things

- The households receipts consist of income from production

$$y_t = f(k_t)$$

- Net transfer payments from the government

$$v_t$$

- They also can use their current money holding to buy things

$$m_t$$

  - as well as their current bond holdings

$$b_t$$

  - as well as their current holdings of capital

$$k_t$$

- Therefore total resources available for buying things are

$$f(k_t) + v_t + m_t + b_t + k_t$$

Purchases and Assets Acquired

- The households can spend on consumption

$$c_t$$

- They also can take part of their resources and use them to acquire money for next period

$$\frac{M_{t+1}}{P_t} = (1 + \pi_t)\, m_{t+1}$$

  - and to acquire bonds for next period

$$\frac{Q_t B_{t+1}}{P_t} = (1 + r_t)^{-1}\, b_{t+1}$$

  - and to acquire capital for next period

$$k_{t+1}$$

- Therefore total uses of resources are

$$c_t + (1 + \pi_t)\, m_{t+1} + (1 + r_t)^{-1}\, b_{t+1} + k_{t+1}$$

Household's Budget Constraint

- Resources available for acquisition in period t are

$$f(k_t) + v_t + m_t + b_t + k_t$$

- Uses of these resources are

$$c_t + (1 + \pi_t) m_{t+1} + (1 + r_t)^{-1} b_{t+1} + k_{t+1}$$

- These imply

$$\begin{aligned} f(k_t) &+ v_t + m_t + b_t + k_t \\ &= c_t + (1 + \pi_t) m_{t+1} + (1 + r_t)^{-1} b_{t+1} + k_{t+1} \end{aligned}$$

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- These imply

$$f(k_t) + v_t + m_t + b_t + k_t$$
$$= c_t + (1 + \pi_t) m_{t+1} + (1 + r_t)^{-1} b_{t+1} + k_{t+1}$$

- which can be written

$$c_t + (1 + \pi_t) m_{t+1} - m_t + (1 + r_t)^{-1} b_{t+1} - b_t + k_{t+1} - k_t$$
$$= f(k_t) + v_t$$

Walras' Law

- The budget constraint for the economy should satisfy the overall feasibility constraint
$$y_t = c_t + k_{t+1} - k_t$$
- Does it?
- Households
$$c_t + (1+\pi_t)\, m_{t+1} - m_t + (1+r_t)^{-1}\, b_{t+1} - b_t + k_{t+1} - k_t = f(k_t) + \upsilon_t$$
- Government
$$\upsilon_t = (1+\pi_t)\, m_{t+1} - m_t + (1+r_t)^{-1}\, b_{t+1} - b_t$$
- Adding the two together to get total spending and receipts in the economy, we find
$$c_t + k_{t+1} - k_t = f(k_t)$$
    - This constraint is indeed satisfied

Equilibrium for Consumers

- Maximization of utility by individual households in competitive markets implies that households maximize utility

$$U_1 = \sum_{t=1}^{\infty} \beta^{t-1} u(c_t, m_t)$$

  - subject to

$$c_t + (1+\pi_t)\, m_{t+1} - m_t + (1+r_t)^{-1}\, b_{t+1} - b_t + k_{t+1} - k_t = f(k_t) + v_t$$

  - with respect to

$$c_t, m_{t+1}, b_{t+1}, k_{t+1}, t = 1, ..., \infty$$

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$$c_t, m_{t+1}, b_{t+1}, k_{t+1}, t = 1, ..., \infty$$

- The first-order conditions are for c_t, m_{t+1} and k_{t+1}

$$\begin{aligned} u_1(c_t, m_t) - \lambda_t &= 0 \\ \beta u_2(c_{t+1}, m_{t+1}) - \lambda_t(1+\pi_t) + \beta\lambda_{t+1} &= 0 \\ -\lambda_t + \beta\lambda_{t+1}\left[f'(k_{t+1}) + 1\right] &= 0 \end{aligned}$$

Equilibrium for Consumers

- There is nothing to guarantee a positive quantity of bonds. For bonds, either

$$b_{t+1} = 0$$

  - or

$$\lambda_{t+1} - \beta^{-1} \lambda_t \left(1 + r_t\right)^{-1} = 0$$

  - or both, so can write as

$$\begin{aligned} \lambda_{t+1} - \beta^{-1} \lambda_t \left(1 + r_t\right)^{-1} &\leq 0 \\ b_{t+1} \left[\lambda_{t+1} - \beta^{-1} \lambda_t \left(1 + r_t\right)^{-1}\right] &= 0 \end{aligned}$$

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- Conditions that are jointly sufficient with these first-order conditions are the transversality conditions

$$\begin{aligned} \lim_{t\to\infty} m_{t+1}\beta^{t-1}\lambda_t (1 + \pi_t) &= 0 \\ \lim_{t\to\infty} k_{t+1}\beta^{t-1}\lambda_t &= 0 \\ \lim_{t\to\infty} b_{t+1}\beta^{t-1}\lambda_t (1 + r_t) &= 0 \end{aligned}$$

Observations about Equilibrium

- $u_1(c_t, m_t) - \lambda_t = 0$ and $u_1(c_t, m_t) > 0$ for all t implies that $\lambda_t > 0$
- If $\lambda_{t+1} = \lambda_t$ (steady state usually),

$$u_1(c_{t+1}, m_{t+1}) = u_1(c_t, m_t)$$

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$$-\lambda_t + \beta\lambda_{t+1}\left[f'(k_{t+1}) + 1\right] = 0$$

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1 + f'(k_{t+1}) &= \beta^{-1}
\end{aligned}$$

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1 + f'(k_{t+1}) &= 1 + \delta
\end{aligned}$$

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Further Observations about Equilibrium

- If $\lambda_{t+1} = \lambda_t$ (steady state usually),

$$u_1\left(c_{t+1}, m_{t+1}\right) = u_1\left(c_t, m_t\right)$$

- with bonds positive, i.e. $b_t > 0$

$$\lambda_{t+1} - \beta^{-1}\lambda_t\left(1 + r_t\right)^{-1} \quad = \quad 0$$

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\beta^{-1} &= \left(1 + r_t\right)
\end{aligned}$$

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\beta^{-1}\left(1 + r_t\right)^{-1} &= 1 \\
\beta^{-1} &= \left(1 + r_t\right) \\
\delta &= r_t = r
\end{aligned}$$

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- For money,

$$\beta u_2\left(c_{t+1}, m_{t+1}\right) - \lambda\left(1 + \pi_t\right) + \beta\lambda = 0$$

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$$\begin{aligned} \beta u_2\left(c_{t+1}, m_{t+1}\right) - \lambda\left(1+\pi_t\right) + \beta\lambda &= 0 \\ u_2\left(c_{t+1}, m_{t+1}\right) - u_1\beta^{-1}\left(1+\pi_t\right) + u_1 &= 0 \end{aligned}$$

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u_2\left(c_{t+1}, m_{t+1}\right) - u_1 \beta^{-1}\left(1+\pi_t\right) + u_1 &= 0 \\
\frac{u_2\left(c_{t+1}, m_{t+1}\right)}{u_1} &= \beta^{-1}\left(1+\pi_t\right) - 1
\end{aligned}$$

Further Observations about Equilibrium

- If $\lambda_{t+1} = \lambda_t$ (steady state usually),

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- For money,

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Summing Up

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Do Yourself

- Set up the model as here
- Then, only looking at the notes when you have to do so, solve the model and determine these properties
- Doing this makes sure you understand the underlying economics and can do the algebra

Optimal Quantity of Money

- Conditions
  - Lump-sum taxes
  - Money printed at zero marginal cost
  - Friedman: The optimal opportunity cost of holding money is zero
  - Stationary state

$$c_t = \bar{c}, k_t = \bar{k}, b_t = \bar{b}$$

$$f'(\bar{k}) = \delta$$

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- Optimal quantity of money

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- Example: Real interest rate is three percent, $r = .03$ implies π=-.02913

Summary

- Have set up and solved an infinite-horizon general equilibrium model with money, bonds and capital
  - Nothing particularly unusual about it
- Does have money in the utility function, which does not reflect any fundamental role of money in the economy
  - Money is just an asset that people like to hold
  - Money used in transactions would be more satisfactory in some ways
  - Money used in transactions as generally implemented is less satisfactory in some ways
- Many general properties of such models shown
- Optimal quantity of money falls out nicely
- McCallum also shows that Ricardian equivalence holds in this model
  - You can study this on your own and make sure you understand it