Financial Econometrics

Multivariate Time Series Analysis: Simultaneous Equations and VARs

Gerald P. Dwyer

Trinity College, Dublin

January 2016

Outline

1. Multivariate Time Series Analysis
   - Structural Equations
   - Vector Autoregression (VAR)
   - Summary

Structural equations

- Suppose have simultaneous system for supply and demand

$$q_t = \alpha^d + \beta^d p_t + \gamma^d S_t + \varepsilon_t^d \qquad (1)$$
$$q_t = \alpha^s + \beta^s p_t + \gamma^s T_t + \varepsilon_t^s$$

  - p is price, q is quantity and S and T are other variables that affect demand and supply
  - Equations jointly determine suppy and demand
  - Called "structural equations

- Suppose solve for "reduced form" with time subscript suppressed

$$q = \pi_{10} + \pi_{11} T + \pi_{12} S + u_1 \qquad (2)$$
$$p = \pi_{20} + \pi_{21} T + \pi_{22} S + u_2$$

Estimation of structural and reduced form equations

- In general, cannot estimate supply and demand structural equations consistently by Ordinary Least Squares (OLS) because p is correlated with error terms in

$$q_t = \alpha^d + \beta^d p_t + \gamma^d S_t + \varepsilon_t^d$$
$$q_t = \alpha^s + \beta^s p_t + \gamma^s T_t + \varepsilon_t^s$$

  - Can't estimate consistently either by reversing equations with p on the left either because q (which would be on the right) is correlated with the error terms also
- Can estimate reduced form regressions by OLS

$$q = \pi_{10} + \pi_{11} T + \pi_{12} S + u_1$$
$$p = \pi_{20} + \pi_{21} T + \pi_{22} S + u_2$$

- Cannot necessarily infer effect of price on quantity demanded from reduced form (although it is possible in this case)
- Constant terms and coefficients of endogenous variables (p and q) and exogenous variables (S and T)

Estimation of structural and reduced form equations

- In general, cannot estimate supply and demand structural equations consistently by Ordinary Least Squares (OLS) because p is correlated with error terms in

$$q_t = \alpha^d + \beta^d p_t + \gamma^d S_t + \varepsilon_t^d$$
$$q_t = \alpha^s + \beta^s p_t + \gamma^s T_t + \varepsilon_t^s$$

- Can estimate reduced form regressions by OLS

$$q = \pi_{10} + \pi_{11} T + \pi_{12} S + u_1$$
$$p = \pi_{20} + \pi_{21} T + \pi_{22} S + u_2$$

- Cannot necessarily infer effect of price on quantity demanded from reduced form (although it is possible in this case)
- Endogenous variables (p and q) and exogenous variables (S and T)
- Same number of constant terms and coefficients in structural equations and reduced form equations
  - Suggests that it may be possible to infer structural coefficients from reduced form coefficients but does not imply it is possible

Exogenous variables and endogenous variables in economics and finance

- In economics and finance theory
  - A variable is **exogenous** if it is determined outside the theory
    - For example, income in Ireland in supply and demand for books in Ireland
    - Level of real income in the world in an equation estimating risk factors for BMW's stock return
  - A variable is **endogenous** if it is determined by the theory
    - For example, quantity of books sold in Ireland is determined by supply and demand for books in Ireland
    - Risk premia for BMW stock in context of Capital-asset pricing model or Arbitrage Pricing Theory

Exogenous variables and endogenous variables in econometrics

- In econometrics
  - A variable is **exogenous** in to an equation if the variable is uncorrelated with current and future error terms
    - For example, S and T in demand and supply earlier
  - A variable is **predetermined** in an equation if its value is uncorrelated with current error term
    - For example, p_{t-1} might affect current quantity demanded if people take time to adjust to changes in price
    - Would say that p_{t-1} is predetermined because it is part of an ordered sequence determined before t
    - It is uncorrelated with ε_t^d and ε_t^s
  - A variable is **endogenous** if it is correlated with the error term in that equation
    - For example, p and q in demand and supply earlier
  - Recall, can get consistent estimates of a set of parameters by OLS if the covariance of the variables with the current and future error terms is zero

Exogenous variables and endogenous variables

- In economics and finance theory
  - A variable is **exogenous** if it is determined outside the theory
  - A variable is **endogenous** if it is determined by the theory
- In econometrics
  - A variable is **exogenous** relative to an equation if the variable is uncorrelated with current and future error terms
  - A variable is **endogenous** if it is correlated with the error term in that equation

Endogenous and exogenous variables

- What is exogenous and what is endogenous in this set of demand and supply equations?

$$q_t = \alpha^d + \beta^d p_t + \gamma^d S_t + \varepsilon_t^d$$
$$q_t = \alpha^s + \beta^s p_t + \gamma^s T_t + \varepsilon_t^s$$

Identification

- Suppose have simultaneous system for supply and demand

$$q_t = \alpha^d + \beta^d p_t + \varepsilon_t^d$$
$$q_t = \alpha^s + \beta^s p_t + \varepsilon_t^s$$

  - No observable variables besides price and quantity
  - Reduced form is

$$q = \pi_{10} + u_1$$
$$p = \pi_{20} + u_2$$

- Two constants and error term
- Cannot possibly infer effect of price on quantity or vice versa

Solutions to identification problem

- Economists have ways of solving the problem
- In fact, the effects of changes in price on quantity supplied and demanded can be estimated from the reduced-form equations sometimes

$$q = \pi_{10} + \pi_{11} T + \pi_{12} S + u_1$$
$$p = \pi_{20} + \pi_{21} T + \pi_{22} S + u_2$$

- For example, in these equations, can infer coefficients in structural equations from the reduced-form equations

All of this can be illustrated with a couple of graphs

- Graphs and estimating structural equation
- Illustrate **rank condition**: The number of exogenous variables excluded from an equation is greater than or equal to the number of endogenous variables in the equation minus 1

Estimation methods

- Simultaneous equations

$$q_t = \alpha^d + \beta^d p_t + \gamma^d S_t + \varepsilon_t^d$$
$$q_t = \alpha^s + \beta^s p_t + \gamma^s T_t + \varepsilon_t^s$$

  - can be estimated by
  - Indirect least squares
    - Use parameters in reduced form to infer coefficients in structual equations above
  - Instrumental variables
    - Find variable correlated with price but not with error term in demand equation (ε_t^d) and not included in demand equation
    - Run reduced-form regression of price on exogenous variables
    - Use predicted value of price in regression for demand equation to get estimated coefficient
    - Similarly supply
    - Here, S and T are such variables
  - Two-stage least squares

Vector autoregressions

- Simple autoregression

$$r_t = \phi_0 + \phi_1 r_{t-1} + \varepsilon_t$$

First-order vector autoregression

- For a vector? First-order vector autoregression is

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \mathbf{\Phi}_1 \mathbf{r}_{t-1} + \mathbf{u}_t$$

  where there are k variables in the vector $\mathbf{r}_t$

$$\mathbf{r}'_t = [r_{1t}, ..., r_{kt}]$$
$$\boldsymbol{\phi}'_0 = [\phi_{10}, ..., \phi_{k0}]$$
$$\mathbf{\Phi}_1 = \begin{bmatrix} \phi_{11} & \dots & \phi_{1k} \\ & \dots & \\ \phi_{k1} & & \phi_{kk} \end{bmatrix},$$
$$\mathbf{u}'_t = [u_{1t}, ..., u_{kt}]$$

  where for ϕ_{ij}, i is equation, j is variable in equation
- Common to put vectors and matrices in bold and I'll do that

Two-variable example

- First-order vector autoregression is

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\Phi}_1 \mathbf{r}_{t-1} + \mathbf{u}_t$$

- For two variables, this is

$$\begin{aligned} r_{1t} &= \phi_{10} + \phi_{11} r_{1,t-1} + \phi_{12} r_{2,t-1} + u_{1,t} \\ r_{2t} &= \phi_{20} + \phi_{21} r_{1,t-1} + \phi_{22} r_{2,t-1} + u_{2,t} \end{aligned}$$

Estimation Strategy

- Common practice is to use Ordinary Least Squares (OLS) to estimate this set of equations
  - Why?
  - Assume that error terms are not correlated with past values of variables
    - ⋆ Can be correlated with future values
  - Same variables in every equation so OLS is equivalent to Seemingly Unrelated Regression
  - VAR is like a reduced form
    - ⋆ No current values of variables on left-hand side of any equation are on the right-hand side of any other equation

Additional Lags

- It also is possible to have a second-order autoregression

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\Phi}_1 \mathbf{r}_{t-1} + \boldsymbol{\Phi}_2 \mathbf{r}_{t-2} + \mathbf{u}_t$$

- or more generally, a kth order autoregression

$$\mathbf{r}_t = \boldsymbol{\phi}_0 + \boldsymbol{\Phi}_1 \mathbf{r}_{t-1} + \boldsymbol{\Phi}_2 \mathbf{r}_{t-2} + \ldots + \boldsymbol{\Phi}_k \mathbf{r}_{t-k} + \mathbf{u}_t$$

Higher-order autoregressions and lag length

- Can write a p-th order autoregression as

$$\mathbf{r}_t = \boldsymbol{\phi}_o + \sum_{i=1}^{p} \boldsymbol{\Phi}_i \mathbf{r}_{t-i} + \mathbf{a}_t$$

- Lag operator can be handy, $\mathrm{L}\,\mathbf{r}_t = \mathbf{r}_{t-1}$ and $\mathrm{L}^i\,\mathbf{r}_t = \mathbf{r}_{t-i}$
- This implies $\boldsymbol{\Phi}_i \mathbf{r}_{t-i} = \boldsymbol{\Phi}_i\,\mathrm{L}^i\,\mathbf{r}_t$ and

$$\sum_{i=1}^{p} \boldsymbol{\Phi}_i \mathbf{r}_{t-i} = \sum_{i=1}^{p} \boldsymbol{\Phi}_i\,\mathrm{L}^{i-1}\,\mathbf{r}_t = \boldsymbol{\Phi}\left(\mathrm{L}\right)\mathbf{r}_{t-1}$$

so

$$\mathbf{r}_t = \boldsymbol{\phi}_o + \boldsymbol{\Phi}\left(\mathrm{L}\right)\mathbf{r}_{t-1} + \mathbf{a}_t$$

- Higher-order autoregression doesn't add anything of substance, at least formally
  - Extra lags do add a lot of parameters

Estimation of higher-order autoregressions

- A fairly common practice in economics is to use the same lag length for all equations for all variables
  - Contributed to Arnold Zellner calling them "Very Awful Regressions"
  - How decide reliably that some coefficients should be zero?
  - Suppose that lags one and four have t-ratios of 3 and lags two and three have t-ratios around one
    - ★ Are the coefficients really zero or just imprecisely estimated?
  - Many extraneous coefficients leads to wide confidence intervals for forecasts
- Determine lag length same way as for univariate autoregressions
  - χ^2 test until five percent "significance"
  - Akaike information criterion
  - (Schwarz) Bayesian information criterion
  - Hannan-Quinn criterion

Bayesian Vector Autoregression

- A Bayesian Vector Autoregression commonly starts from a "Minnesota prior": The prior supposes that each series is a random walk with nonzero probability on additional lags
  - Data then update this prior
  - Improves precision of estimated coefficients and, to some degree, starts with a simple characterization

Reduced form and structural equations

- Can interpret a VAR as a reduced form of some equation system
  - Two-variable VAR for money and the price level

$$\begin{aligned} m_t &= \phi_{10} + \phi_{11} m_{t-1} + \phi_{12} p_{t-1} + a_{1t} \\ p_t &= \phi_{20} + \phi_{21} m_{t-1} + \phi_{22} p_{t-1} + a_{2t} \end{aligned} \tag{3}$$

  - This has no simultaneous determination
- Suppose economic theory suggests

$$\begin{aligned} m_t &= \alpha_m + \beta_{1,11} m_{t-1} + \beta_{1,20} p_t + \beta_{1,21} p_{t-1} + \varepsilon_t^m \\ p_t &= \alpha_p + \beta_{2,10} m_t + \beta_{2,11} m_{t-1} + \beta_{2,21} p_{t-1} + \varepsilon_t^p \end{aligned}$$

  - $\{\varepsilon_t^m\}$ and $\{\varepsilon_t^p\}$ are zero mean, constant variance, serially uncorrelated processes
  - $\beta_{1,20}$ may reflect effect of prices on the nominal quantity of money demanded
  - $\beta_{2,10}$ may reflect effect of supply of money on prices
- Can solve for reduced form and it will look exactly like the VAR reduced form (3)

Recursive version

- Inclusion of contemporary variable in only one of two equations (**recursive system**)

$$m_t = b_{10} + b_{11} m_{t-1} + b^* p_t + b_{12} p_{t-1} + e_t^m$$
$$p_t = \phi_{20} + \phi_{21} m_{t-1} + \phi_{22} p_{t-1} + e_t^p$$

  - Estimation of both equations by OLS is consistent if the errors in the two equations are uncorrelated
  - Can compute from OLS estimation of VAR using Cholesky decomposition

- How do these coefficients relate to **simultaneous system**

$$m_t = \alpha_m + \beta_{1,11} m_{t-1} + \beta_{1,20} p_t + \beta_{1,21} p_{t-1} + \varepsilon_t^m$$
$$p_t = \alpha_p + \beta_{2,10} m_t + \beta_{2,11} m_{t-1} + \beta_{2,21} p_{t-1} + \varepsilon_t^p$$

or to reduced form **VAR**

$$m_t = \phi_{10} + \phi_{11} m_{t-1} + \phi_{12} p_{t-1} + \varepsilon_t^m$$
$$p_t = \phi_{20} + \phi_{21} m_{t-1} + \phi_{22} p_{t-1} + \varepsilon_t^p$$

Relationship between recursive system, simultaneous system and VAR

- Clearly recursive system, simultaneous system and VAR do not represent the same things
- VAR is a reduced form of simultaneous system
  - Coefficients in VAR are functions of the coefficients in the simultaneous system
  - Error terms in VAR are a combination of the error terms in the simultaneous system
- Recursive system is a version of the VAR in which all of the correlation between ε_t^m and ε_t^p in the VAR is reflected in the coefficient of p_t in the m_t equation in the recursive system
- There is no necessary reason that the coefficients in the simultaneous system should equal the coefficients in the recursive system
  - The coefficients will be equal if the behavioral simultaneous system is recursive
    - The coefficient of m_t in the p_t equation is zero
    - And the errors are uncorrelated
    - Not otherwise

Granger causality tests and VARs

- Relationship of money growth and inflation

$$\begin{aligned} m_t &= \phi_{10} + \phi_{11} m_{t-1} + \phi_{12} p_{t-1} + \varepsilon_t^m \\ p_t &= \phi_{20} + \phi_{21} m_{t-1} + \phi_{22} p_{t-1} + \varepsilon_t^p \end{aligned} \tag{4}$$

- Does money "cause" inflation?
- One version of "cause": Granger causality
  - Money causes inflation if and only if the equations for money and inflation (4) **cannot** be rewritten

$$\begin{aligned} m_t &= \phi_{10} + \phi_{11} m_{t-1} + \phi_{12} p_{t-1} + \varepsilon_t^m \\ p_t &= \phi_{20} + \phi_{22} p_{t-1} + \varepsilon_t^p \end{aligned} \tag{5}$$

  - If money helps to predict inflation, then money "causes" inflation, $\phi_{21} \neq 0$, equations (4)
  - If money does not help to predict inflation, then money does "not cause" inflation, $\phi_{21} = 0$, equations (4) reduce to equations (5)

Granger cauasality and multiple lags

- Use F-tests

$$
\begin{aligned}
m_t &= \phi_{10} + \phi_{11} m_{t-1} + \phi_{12} p_{t-1} + \varepsilon_t^m \\
p_t &= \phi_{20} + \phi_{21} m_{t-1} + \phi_{21,2} m_{t-2} + \phi_{22} p_{t-1} + \varepsilon_t^p
\end{aligned} \tag{6}
$$

  - Test whether both coefficients of lagged money are statistically significant to test whether money helps to predict inflation
    - $\phi_{21} = \phi_{21,2} = 0$
    - F-test on both coefficients jointly

Granger causality and test on more than one equation at a time

- Joint test across equations on

$$m_t = \phi_{10} + \phi_{11} m_{t-1} + \phi_{12} p_{t-1} + \phi_{13} y_{t-1} + \varepsilon_t^m$$
$$p_t = \phi_{20} + \phi_{21} m_{t-1} + \phi_{22} p_{t-1} + \phi_{23} y_{t-1} + \varepsilon_t^p$$
$$y_t = \phi_{20} + \phi_{31} m_{t-1} + \phi_{32} p_{t-1} + \phi_{33} y_{t-1} + \varepsilon_t^y$$

  - Want to test whether money helps to predict p and y
    - Test whether coefficients of lagged money are statistically significant, $\phi_{21} = \phi_{31} = 0$
    - F-test or χ^2 test on both coefficients jointly (uses determinant of covariance matrix instead of sum of squared residuals)

Exogeneity

- Granger causality is related to exogeneity
  - For many practical econometric purposes, the issue is whether the variable can be included in the right-hand side of an equation and one can ignore how it is determined
  - Generally can ignore variable's determination if the theoretical covariance of the variable with the error term is zero

Example of Granger causality and structural equations

- Relationship between money and inflation – Suppose economic theory suggests

$$m_t = \alpha_m + \beta_{1,11} m_{t-1} + \beta_{1,20} p_t + \beta_{1,21} p_{t-1} + \varepsilon_t^m$$
$$p_t = \alpha_p + \beta_{2,10} m_t + \beta_{2,11} m_{t-1} + \beta_{2,21} p_{t-1} + \varepsilon_t^p$$

- Two-variable VAR for money and the price level

$$m_t = \phi_{10} + \phi_{11} m_{t-1} + \phi_{12} p_{t-1} + a_{1t}$$
$$p_t = \phi_{20} + \phi_{21} m_{t-1} + \phi_{22} p_{t-1} + a_{2t}$$

- If money helps to predict inflation but inflation does not help to predict money, $\phi_{12} = 0$, and $\beta_{1,20} = \beta_{1,21} = 0$, and money Granger causes inflation
  - Then can write

$$m_t = a_m + b_{1,11} m_{t-1} + \varepsilon_t^m$$
$$p_t = a_p + b_{2,10} m_t + b_{2,11} m_{t-1} + b_{2,21} p_{t-1} + \varepsilon_t^p$$

  - It is possible to write structural equations with money exogenous relative to inflation

Impulse responses and Variance decompositions

- Impulse responses trace out the effect of a shock to one variable, ε_0^m, in one period, e.g. period 0, on all the variables
- Variance decompositions show how much of the variance of each variable is due to shocks to each variable, e.g. ε_0^m and ε_0^p

Summary

- Multiple equations allow for interactions to be represented
- Simultaneous equations can be estimated by instrumental variables
- Vector autogressions are similar to autoregressions but include more than one variable
- Structural equations
- Granger Causality
- Impulse response functions