Multivariate Time Series: Part 2

Simultaneous Equations and Vector Autoregressions

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Outline

Identification

- We have seen that statistical fit will not allow allow us to resolve identification issues
- Maybe Economics can help us out

Structural equations

- Suppose have simultaneous system for supply and demand

$$
\begin{aligned}
q_t^d &= \alpha p_t + \beta y_t + \varepsilon_t^d, \, \alpha < 0, \beta > 0 \\
q_t^s &= \gamma p_t + \delta w_t + \varepsilon_t^s, \;\; \gamma, \delta > 0 \\
q_t &= q_t^d = q_t^s
\end{aligned} \tag{1}
$$

  - p is price, q is quantity and y and w are other variables that affect demand and supply
  - Suppose $\mathrm{E}\, \varepsilon_t^d = \mathrm{E}\, \varepsilon_t^s = 0$, $\mathrm{E}\, \varepsilon_t^d \varepsilon_t^s = 0$, constant variances and both serially uncorrelated)
  - For convenience, suppose that all variables have expected values of zero
  - Equations jointly determine quantities bought and sold as well as price

Structural equations

- Simultaneous system for supply and demand

$$
\begin{aligned}
q_t^d &= \alpha p_t + \beta y_t + \varepsilon_t^d,\ \alpha < 0, \beta > 0 \\
q_t^s &= \gamma p_t + \delta w_t + \varepsilon_t^s,\ \ \gamma, \delta > 0 \\
q_t &= q_t^d = q_t^s
\end{aligned}
$$

- Why are they called structural equations?
- Called **structural equations** because they summarize the behavior of economic agents
  - Demand equation summarizes behavior of buyers and is invariant to changes in supply
  - Supply equation summarizes behavior of sellers and is invariant to changes in demand

Structural equations

- Have simultaneous system for supply and demand

$$q_t = \alpha p_t + \beta y_t + \varepsilon_t^d,\ \alpha < 0, \beta > 0$$
$$q_t = \gamma p_t + \delta w_t + \varepsilon_t^s,\ \ \gamma, \delta > 0$$

- An equation is **structural** relative to an intervention if it is invariant to that intervention
  - Example: Suppose a sales tax is imposed on purchases and the demand equation is changed to include $(1+\tau)\,p$ where τ is the sales tax
  - Then these equations are structural relative to that intervention

Structural equations

- Simultaneous system for supply and demand

$$q_t = \alpha p_t + \beta y_t + \varepsilon_t^d,\ \alpha < 0, \beta > 0$$
$$q_t = \gamma p_t + \delta w_t + \varepsilon_t^s,\ \ \gamma, \delta > 0$$

- The variables p and q are **jointly dependent**
- The variables y and w are **exogenous**
  - In the context of theories, this means a variable is "determined outside the system being examined"
  - In econometric contexts, it can mean a variable is "uncorrelated with the error term" in this and possibly other equations

Structural equations and reduced form

- Simultaneous system for supply and demand

$$q_t = \alpha p_t + \beta y_t + \varepsilon_t^d, \; \alpha < 0, \beta > 0$$
$$q_t = \gamma p_t + \delta w_t + \varepsilon_t^s, \;\; \gamma, \delta > 0$$

- Can solve for **reduced form** (with time subscript suppressed)

$$q = (\gamma - \alpha)^{-1} \left[\gamma\beta y - \alpha\delta w + \gamma\varepsilon_t^d - \alpha\varepsilon_t^s \right]$$
$$p = (\gamma - \alpha)^{-1} \left[\beta y - \delta w + \varepsilon_t^d - \varepsilon_t^s \right]$$

  - Cramer's rule is the simplest way to do it, at least for me

Reduced form

- Reduced form is

$$q = (\gamma - \alpha)^{-1} \left[\gamma\beta y - \alpha\delta w + \gamma\varepsilon_t^d - \alpha\varepsilon_t^s \right]$$
$$p = (\gamma - \alpha)^{-1} \left[\beta y - \delta w + \varepsilon_t^d - \varepsilon_t^s \right]$$

- This also can be written

$$\begin{aligned} q &= \pi_{11} y + \pi_{12} w + u_1 \\ p &= \pi_{21} y + \pi_{22} w + u_2 \end{aligned} \tag{2}$$

where

$$\pi_{11} = (\gamma - \alpha)^{-1} \gamma\beta$$
$$\pi_{12} = -(\gamma - \alpha)^{-1} \alpha\delta$$
$$\pi_{21} = (\gamma - \alpha)^{-1} \beta$$
$$\pi_{22} = -(\gamma - \alpha)^{-1} \delta$$

Reduced form

- In this example with

$$\pi_{11} = (\gamma - \alpha)^{-1} \gamma\beta$$
$$\pi_{12} = -(\gamma - \alpha)^{-1} \alpha\delta$$
$$\pi_{21} = (\gamma - \alpha)^{-1} \beta$$
$$\pi_{22} = -(\gamma - \alpha)^{-1} \delta$$

- It is easy to show that given π_{ij}, we can solve for

$$\alpha = \pi_{12}/\pi_{22}$$
$$\gamma = \pi_{11}/\pi_{21}$$
$$\beta = (\gamma - \alpha)\,\pi_{21}$$
$$\delta = (\gamma - \alpha)\,\pi_{22}$$

if π_{21} and π_{22} are not equal to zero

Identification and estimation

- In this example with

$$q_t = \alpha p_t + \beta y_t + \varepsilon_t^d,\ \alpha < 0, \beta > 0$$
$$q_t = \gamma p_t + \delta w_t + \varepsilon_t^s,\ \ \gamma, \delta > 0$$

- We can estimate the parameters by estimating the reduced form by OLS and then inferring the structural parameters
- This is called **indirect least squares**
  - Not generally available but sometimes is
  - With simple equations here with serially uncorrelated errors and no lagged variables, can estimate reduced form by ordinary least squares
  - Given the estimated reduced form, solve for the structural coefficients

Indirect Least Squares

- How Indirect Least Squares works
  - Estimate reduced form by OLS and get estimates

$$\begin{array}{cc} \widehat{\pi}_{11} & \widehat{\pi}_{12} \\ \widehat{\pi}_{21} & \widehat{\pi}_{22} \end{array}$$

  - These coefficients are estimated consistently because the error terms are uncorrelated with the right-hand-side variables
- Furthermore if π_{21} and π_{22} are not equal to zero

$$\widehat{\alpha} = \widehat{\pi}_{12} / \widehat{\pi}_{22}$$
$$\widehat{\gamma} = \widehat{\pi}_{11} / \widehat{\pi}_{21}$$
$$\widehat{\beta} = (\widehat{\gamma} - \widehat{\alpha})\, \widehat{\pi}_{21}$$
$$\widehat{\delta} = (\widehat{\gamma} - \widehat{\alpha})\, \widehat{\pi}_{22}$$

  - and these are consistent estimators of α, γ, β, and δ
    - ★ Recall that $\text{plim}\, ab = \text{plim}\, a\, \text{plim}\, b$

OLS and estimation of structural equations

- Why can't we just estimate the structural equations by OLS?
- Suppose estimate this demand equation by OLS
  - Simplify algebra by supposing $\beta = 0$ so we have

$$q_t = \alpha p_t + \varepsilon_t^d,\ \alpha < 0$$

  - Estimate α by

$$\widehat{\alpha} = \frac{\sum qp}{\sum p^2}$$

  - What is the equation for the quantity q? It is

$$q_t = \alpha p_t + \varepsilon_t^d$$

OLS and estimation of structural equations

- Substitute this equation for q into the OLS estimating equation to get

$$\begin{aligned}\widehat{\alpha} &= \frac{\sum\left(\alpha p_t + \varepsilon_t^d\right) p}{\sum p^2} \\ &= \alpha \frac{\sum p^2}{\sum p^2} + \frac{\sum \varepsilon_t^d p}{\sum p^2} \\ &= \alpha + \frac{\sum \varepsilon_t^d p}{\sum p^2}\end{aligned}$$

- Question of consistency comes down to whether

$$\text{plim} \frac{\sum \varepsilon_t^d p / T}{\sum p^2 / T} = 0?$$

OLS and estimation of structural equations

- Question of consistency comes down to whether

$$\text{plim} \frac{\sum \varepsilon_t^d p / T}{\sum p^2 / T} = \frac{\text{plim} \sum \varepsilon_t^d p / T}{\text{plim} \sum p^2 / T} = 0?$$

- But it is not. The price is

$$p = (\gamma - \alpha)^{-1} \left[-\delta w + \varepsilon_t^d - \varepsilon_t^s \right]$$

- And

$$\text{plim} \sum \varepsilon_t^d (\gamma - \alpha)^{-1} \left[-\delta w + \varepsilon_t^d - \varepsilon_t^s \right] / T \neq 0$$

  in general because ε_t^d affects both the price and the quantity demanded

- This is a system of equations that simultaneously determine price and quantity

Structural estimation methods

- Structural estimation methods come down to finding estimators that either directly consider this correlation or else circumvent it in a way similar to instrumental variables

Structural equations and VARs

- Suppose have dynamic simultaneous system for supply and demand

$$q_t = [\alpha_0 + \alpha(\mathsf{L})]\, p_t + \alpha^*(\mathsf{L})\, q_{t-1} + \beta y_t + \varepsilon_t^d$$
$$q_t = [\gamma_0 + \gamma(\mathsf{L})]\, p_t + \gamma^*(\mathsf{L})\, q_{t-1} + \delta w_t + \varepsilon_t^s$$

- where

$$\alpha_0 + \alpha(\mathsf{L}) = \alpha_0 + \sum_{i=1}^{k} \alpha_i \mathsf{L}^i \qquad \alpha^*(\mathsf{L}) = \sum_{i=1}^{k} \alpha^{*i} \mathsf{L}^i$$
$$\gamma_0 + \gamma(\mathsf{L}) = \gamma_0 + \sum_{i=1}^{k} \gamma_i \mathsf{L}^i \qquad \gamma^*(\mathsf{L}) = \sum_{i=1}^{k} \gamma^{*i} \mathsf{L}^i$$

- Now solve for reduced form
- The lagged values are similar to exogenous variables, i.e. not solved out
- If the lagged values are uncorrelated with the error terms, they can be called **predetermined variables**

Structural equations and VARs

- Solve for reduced form

$$q_t = \Pi_{11}(\mathsf{L})\, q_{t-1} + \Pi_{12}(\mathsf{L})\, p_{t-1} + \pi_{11} y_t + \pi_{12} w_t + u_{t1}$$
$$p_t = \Pi_{21}(\mathsf{L})\, q_{t-1} + \Pi_{22}(\mathsf{L})\, p_{t-1} + \pi_{21} y_t + \pi_{22} w_t + u_{t2}$$

- where $\Pi_{11}(\mathsf{L}) = \sum_{i=1}^{k} \Pi_{11,i}\, \mathsf{L}^{i-1}$, and so forth
- Similar to a vector autoregression except have variables y and w
- This is a vector autoregression VAR if have one of the following
  - y and w have coefficients of zero
  - y and w are serially uncorrelated and we sweep them into the error terms
  - we expand this into a 4-variable VAR
    - Maybe with just simple autoregressions for y and w if there is no reason to think that y and w depend on past prices and quantities

Structural equations and VARs

- Two-variable VAR

$$
\begin{aligned}
q_t &= \Pi_{11}(L)\, q_{t-1} + \Pi_{12}(L)\, p_{t-1} + u_{t1} \\
p_t &= \Pi_{21}(L)\, q_{t-1} + \Pi_{22}(L)\, p_{t-1} + u_{t2}
\end{aligned}
$$

- Possible four-variable set of equations

$$
\begin{aligned}
q_t &= \Pi_{11}(L)\, q_{t-1} + \Pi_{12}(L)\, p_{t-1} + \pi_{11} y_t + \pi_{12} w_t + u_{t1} \\
p_t &= \Pi_{21}(L)\, q_{t-1} + \Pi_{22}(L)\, p_{t-1} + \pi_{21} y_t + \pi_{22} w_t + u_{t2} \\
y_t &= \Pi_{33}(L)\, y_{t-1} + u_{t3} \\
w_t &= \Pi_{34}(L)\, w_{t-1} + u_{t4}
\end{aligned}
$$

- Possible four-variable VAR

$$
\begin{aligned}
q_t &= \Pi_{11}(L)\, q_{t-1} + \Pi_{12}(L)\, p_{t-1} + \Pi_{31}(L)\, y_{t-1} + \Pi_{41}(L)\, w_{t-1} + u_{t1} \\
p_t &= \Pi_{21}(L)\, q_{t-1} + \Pi_{22}(L)\, p_{t-1} + \Pi_{32}(L)\, y_{t-1} + \Pi_{42}(L)\, w_{t-1} + u_{t2} \\
y_t &= \Pi_{33}(L)\, y_{t-1} + u_{t3} \\
w_t &= \Pi_{34}(L)\, w_{t-1} + u_{t4}
\end{aligned}
$$

Structural equations and VARs

- In reduced form

$$
\begin{aligned}
q_t &= \Pi_{11}(\mathsf{L})\, q_{t-1} + \Pi_{12}(\mathsf{L})\, p_{t-1} + \Pi_{31}(\mathsf{L})\, y_{t-1} + \Pi_{41}(\mathsf{L})\, w_{t-1} + u_{t1} \\
p_t &= \Pi_{21}(\mathsf{L})\, q_{t-1} + \Pi_{22}(\mathsf{L})\, p_{t-1} + \Pi_{32}(\mathsf{L})\, y_{t-1} + \Pi_{42}(\mathsf{L})\, w_{t-1} + u_{t2} \\
y_t &= \Pi_{33}(\mathsf{L})\, y_{t-1} + u_{t3} \\
w_t &= \Pi_{34}(\mathsf{L})\, w_{t-1} + u_{t4}
\end{aligned}
$$

- The error terms u_1 and u_2 in the equations for p and q in the VAR are sums of the innovations in both demand and supply
  - Do not represent demand or supply shocks (innovations) alone
  - Both error terms include both demand and supply shocks

Structural equations and VARs

- Bottom line:
- A VAR can be interpreted as a reduced form of structural equations

Structural equations and recursive VARs

- A recursive system is one in which there the errors are uncorrelated and there is an ordering of variables in equations
  - As below, p_t in q_t equation but not q_t in p_t equation
- Unless the structural equations themselves are *recursive*, the VAR reduced form generally will **not** look like

$$q_t = \pi_0 p_t + \Pi_{11}(\mathsf{L})\, q_{t-1} + \Pi_{12}(\mathsf{L})\, p_{t-1} + \pi_{11} y_t + \pi_{12} w_t + u_{1t}$$
$$p_t = \Pi_{21}(\mathsf{L})\, q_{t-1} + \Pi_{22}(\mathsf{L})\, p_{t-1} + \pi_{21} y_t + \pi_{22} w_t + u_{2t}$$

- Estimating such an equation estimates neither the reduced form nor the structural equations
- Does not estimate demand equation or supply equation
- Errors are not demand shocks or supply shocks
- Errors are uncorrelated
  - It is a representation of the relationship between p_t and q_t, just not demand equation, supply equation or reduced form

Government expenditures and real GDP

- Does this matter?
- Government expenditures and real GDP from Andrew Young "Why in the world are we all Keynesians now?", page 12
  - Graphs show response of real GDP to a one-standard deviation "shock" to government spending
  - Top graph has current government expenditures appearing in the real GDP equation

$$y_t = a_1^* g_t + a_{1,1} y_{t-1} + a_{1,2} g_{t-1} + \varepsilon_{1,t}$$
$$g_t = a_{2,1} y_{t-1} + a_{2,2} g_{t-1} + \varepsilon_{2,t}$$

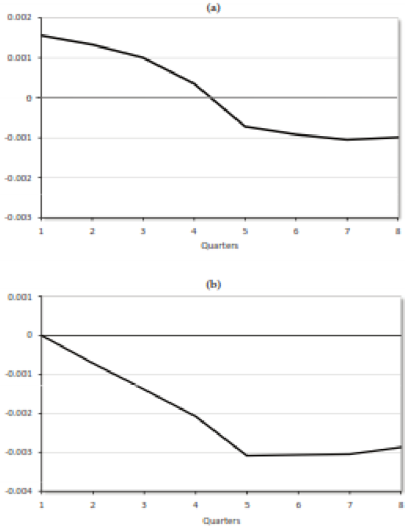
    - ★ Effect of one-standard-deviation change of $\varepsilon_{2,t}$ on y_t
  - Bottom graph has current GDP appearing in the current government expenditures equation

$$y_t = a_{1,1} y_{t-1} + a_{1,2} g_{t-1} + \eta_{1,t}$$
$$g_t = a_2^* g_t + a_{2,1} y_{t-1} + a_{2,2} g_{t-1} + \eta_{2,t}$$

    - ★ Effect of one-standard-deviation change of $\eta_{2,t}$ on y_t

Government expenditures and real GDP

Figure 4
Estimated Effects of Real GDP to a Federal Government Expenditures Shock



Why not estimate simple VAR?

- Why not use

$$y_t = a_{1,1} y_{t-1} + a_{1,2} g_{t-1} + u_{1,t}$$
$$g_t = a_{2,1} y_{t-1} + a_{2,2} g_{t-1} + u_{2,t}$$

- Effect of one-standard-deviation change of $u_{2,t}$ on $y_{1,t}$ doesn't represent effect of change in government's desired spending alone because the error term generally is a function of all structural errors

Summary

- Economics goes a long way in identification problems
- If you don't have an economic theory for a system you are estimating, you are in trouble
- Simultaneous equations systems are very useful when more than one variable is simultaneously determined
- Vector autoregressions can be interpreted as reduced forms of simultaneous equation systems
- Recursive systems of variables are poor – and potentially uninformative or misleading – representations of the relationships among variables