Multivariate Time Series

Introduction: Time Series

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Outline

Vector autoregression

- First-order vector autoregression with n variables

$$\mathbf{y}_t = \mathbf{A}_0 + \mathbf{A}_1 \mathbf{y}_{t-1} + \varepsilon_t$$

- I will indicate vectors and matrices by putting them in bold text
- Variables are in $\mathbf{y}'_t = [y_{1,t}, y_{2,t}, ..., y_{n,t}]$
- $\mathbf{y}_t$ is an $n \times 1$ vector
- $\mathbf{A}_0$ is an $n \times 1$ vector as well
- $\mathbf{A}_1$ must be $n \times n$ since $\mathbf{y}_{t-1}$ is an $n \times 1$ vector and $\mathbf{A}_1$ is post-multiplied by $\mathbf{y}_{t-1}$
- Therefore $\mathbf{A}_1 \mathbf{y}_{t-1}$ is $n \times 1$
- And ε_t is $n \times 1$
- ε_t has zero mean, constant variance and is serially uncorrelated

Vector autoregression written out

- Written out explicitly, this first-order autoregression is

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \\ \dots \\ y_{n,t} \end{bmatrix} = \begin{bmatrix} a_{1,0} \\ a_{2,0} \\ \dots \\ a_{n,0} \end{bmatrix} + \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \dots & \dots & \dots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \\ \dots \\ y_{n,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \dots \\ \varepsilon_{n,t} \end{bmatrix}$$

Vector autoregression written out

- Written out in terms of equations, this first-order autoregression is

$$
\begin{aligned}
y_{1,t} &= a_{1,0} + a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \ldots + a_{1,n}y_{n,t-1} + \varepsilon_{1,t} \\
y_{2,t} &= a_{2,0} + a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \ldots + a_{2,n}y_{n,t-1} + \varepsilon_{2,t} \\
&\ldots \\
y_{n,t} &= a_{n,0} + a_{n,1}y_{1,t-1} + a_{n,2}y_{2,t-1} + \ldots + a_{n,n}y_{n,t-1} + \varepsilon_{n,t}
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\end{aligned}
$$

- Everything depends on lagged values of everything

With more than one lag

- Vector autoregression with more lags

$$\mathbf{y}_t = \mathbf{A}_0 + \mathbf{A}\left(\mathsf{L}\right)\mathbf{y}_{t-1} + \varepsilon_t$$

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- Can also write it as

$$\mathbf{y}_t = \mathbf{A}_0 + \sum_{i=1}^{k} \mathbf{A}_i \mathbf{y}_{t-i} + \boldsymbol{\varepsilon}_t$$

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- Suppose

$$\begin{aligned} \mathrm{E}\,\boldsymbol{\varepsilon}_t &= \mathbf{0} \\ \mathrm{E}\,\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_t' &= \boldsymbol{\Sigma}, \\ \mathrm{E}\,\boldsymbol{\varepsilon}_t \boldsymbol{\varepsilon}_s' &= \mathbf{0} \;\forall\; t \neq s \end{aligned}$$

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- This is a very general representation in terms of the relationships of series

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where $\mathbf{B}\left(\mathrm{L}\right)$ is a matrix of lag polynomials of arbitrary length with

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- Can also write this as

$$\mathbf{y}_t = \sum_{i=0}^{\infty} \mathbf{B}_i\varepsilon_{t-i}$$

MA and AR representation

- How are the moving average and autoregressive representations related?

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- The autoregressive representation is

$$\mathbf{y}_t = \mathbf{A}_0 + \mathbf{A}\left(\mathsf{L}\right)\mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t$$

- If $\mathbf{B}\left(\mathsf{L}\right)$ is invertible, then

$$\begin{aligned} \mathbf{y}_t &= \mathbf{B}_0 + \mathbf{B}\left(\mathsf{L}\right)\boldsymbol{\varepsilon}_t \\ \mathbf{B}\left(\mathsf{L}\right)^{-1}\mathbf{y}_t &= \mathbf{B}\left(\mathsf{L}\right)^{-1}\mathbf{B}_0 + \boldsymbol{\varepsilon}_t \end{aligned}$$

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$$\mathbf{B}\left(\mathsf{L}\right)^{-1}\mathbf{y}_t = \mathbf{B}\left(\mathsf{L}\right)^{-1}\mathbf{B}_0 + \varepsilon_t$$

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- Simplify by looking at univariate case with one lag

$$B(\mathrm{L})^{-1} = \frac{1}{1 + b_1 \mathrm{L}}$$

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$$B(\mathsf{L})^{-1} = \frac{1}{1 + b_1 \mathsf{L}}$$

- What does this equation represent?

Inverse of polynomial in lag operator

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- For the univariate case with one lag

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- Know for a geometric series with an initial value of one and a common ratio r that

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- Know for a geometric series with an initial value of one and a common ratio r that

$$\frac{1}{1 - r} = \sum_{i=0}^{\infty} r^i$$

- Therefore

$$B(\mathrm{L})^{-1} = \sum_{i=0}^{\infty} (-b_1 \mathrm{L})^i$$

Inverse of polynomial in lag operator

- Therefore, for an MA(1), the autoregressive representation is given by

$$
\begin{aligned}
y_t &= b_0 + \varepsilon_t + b_1 \varepsilon_{t-1} \\
\sum_{i=0}^{\infty} (-b_1 \mathrm{L})^i \, y_t &= \sum_{i=0}^{\infty} (-b_1 \mathrm{L})^i \, b_0 + \varepsilon_t \\
y_t &= b_0 \sum_{i=0}^{\infty} (-b_1)^i - \sum_{i=1}^{\infty} (-b_1)^i \, y_{t-i} + \varepsilon_t
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- The general MA representation can generate an AR representation with serially uncorrelated errors
- In general, requires an infinite number of lags in the autoregressive representation

Normalization issues arise

- In the univariate case, we assume that the coefficient of the variable's current own innovation is unity
- Why? Suppose we didn't. Moving-average representation is

$$y_t = \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$$

- Suppose that all $\psi_i = 0 \; \forall \; i > 0$. So

$$y_t = \psi_0 \varepsilon_t$$

Normalization in moving average

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- We can set $\psi_0 = 1$ and then $\mathrm{Var}\left[\varepsilon_t\right] = \mathrm{Var}\left[y_t\right]$

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- We could set $\psi_0^2 = 0.5$ and then $\mathrm{Var}\left[\varepsilon_t\right] = 2\,\mathrm{Var}\left[y_t\right]$

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- Either ψ_0 or $\mathrm{Var}\left[\varepsilon_t\right]$ is not identified by the single number $\mathrm{Var}\left[y_t\right]$
- Setting $\psi_0 = 1$ generally most "natural" but it is arbitrary

Normalization in vector moving average

- In the multivariate case, the representation is

$$\mathbf{y}_t = \mathbf{B}\left(\mathsf{L}\right)\boldsymbol{\varepsilon}_t$$

- Take simpler two-variable case

$$y_{1,t} = \sum_{i=0}^{\infty} \psi_{11,i}\varepsilon_{1,t-i} + \sum_{i=0}^{\infty} \psi_{12,i}\varepsilon_{2,t-i}$$
$$y_{2,t} = \sum_{i=0}^{\infty} \psi_{21,i}\varepsilon_{1,t-i} + \sum_{i=0}^{\infty} \psi_{22,i}\varepsilon_{2,t-i}$$

- Nothing fundamental will be lost if we simplify to no lags of each innovation

$$y_{1,t} = \psi_{11,0}\varepsilon_{1,t} + \psi_{12,0}\varepsilon_{2,t}$$
$$y_{2,t} = \psi_{21,0}\varepsilon_{1,t} + \psi_{22,0}\varepsilon_{2,t}$$

Normalization of zero-order vector moving average

- What are the observable summary statistics?

$$\mathrm{E}\, y_1 \text{ and } \mathrm{E}\, y_2$$

- These just identify constant terms so we will assume

$$\mathrm{E}\, y_1 = \mathrm{E}\, y_2 = 0$$

- In terms of second moments, we have

$$\mathrm{Var}\left[y_1\right],\ \mathrm{Var}\left[y_2\right] \text{ and } \mathrm{Cov}\left[y_1, y_2\right]$$

- By analogy with the univariate case, we might expect that we can identify three parameters and others can be determined by normalizations

Normalization of zero-order vector moving average

- It seems plausible that we may be able to identify three parameters from the data on three moments of variable
- What parameters do we have?

$$y_{1,t} = \psi_{11}\varepsilon_{1,t} + \psi_{12}\varepsilon_{2,t}$$
$$y_{2,t} = \psi_{21}\varepsilon_{1,t} + \psi_{22}\varepsilon_{2,t}$$

- We have

$$\psi_{11} \;\; \psi_{12} \;\; \psi_{21} \;\; \psi_{22}$$

and

$$\text{Var}\left[\varepsilon_1\right], \;\; \text{Var}\left[\varepsilon_2\right] \;\; \text{and} \;\; \text{Cov}\left[\varepsilon_1, \varepsilon_2\right]$$

- Seven parameters and three moments with which we might identify them

Seven parameters and three moments

- The equations are

$$
\begin{aligned}
y_{1,t} &= \psi_{11}\varepsilon_{1,t} + \psi_{12}\varepsilon_{2,t} \\
y_{2,t} &= \psi_{21}\varepsilon_{1,t} + \psi_{22}\varepsilon_{2,t}
\end{aligned}
$$

- The variances and covariances in terms of this moving-average representation are

$$
\begin{aligned}
\text{Var}\left[y_1\right] &= \psi_{11}^2 \text{Var}\left[\varepsilon_1\right] + \psi_{12}^2 \text{Var}\left[\varepsilon_2\right] + 2\psi_{11}\psi_{12} \text{Cov}\left[\varepsilon_1, \varepsilon_2\right] \\
\text{Var}\left[y_2\right] &= \psi_{21}^2 \text{Var}\left[\varepsilon_1\right] + \psi_{22}^2 \text{Var}\left[\varepsilon_2\right] + 2\psi_{21}\psi_{22} \text{Cov}\left[\varepsilon_1, \varepsilon_2\right] \\
\text{Cov}\left[y_1, y_2\right] &= \psi_{11}\psi_{21} \text{Var}\left[\varepsilon_1\right] + \psi_{12}\psi_{22} \text{Var}\left[\varepsilon_2\right] \\
&\quad + \left(\psi_{11}\psi_{22} + \psi_{12}\psi_{21}\right) \text{Cov}\left[\varepsilon_1, \varepsilon_2\right]
\end{aligned}
$$

- It seems plausible that we may be able to identify three parameters from the data on three moments

What normalizations and how can they be interpreted?

- The variances and covariances in terms of this moving-average representation are

$$\begin{aligned}
\text{Var}[y_1] &= \psi_{11}^2 \text{Var}[\varepsilon_1] + \psi_{12}^2 \text{Var}[\varepsilon_2] + 2\psi_{11}\psi_{12} \text{Cov}[\varepsilon_1, \varepsilon_2] \\
\text{Var}[y_2] &= \psi_{21}^2 \text{Var}[\varepsilon_1] + \psi_{22}^2 \text{Var}[\varepsilon_2] + 2\psi_{21}\psi_{22} \text{Cov}[\varepsilon_1, \varepsilon_2] \\
\text{Cov}[y_1, y_2] &= \psi_{11}\psi_{21} \text{Var}[\varepsilon_1] + \psi_{12}\psi_{22} \text{Var}[\varepsilon_2] \\
&\quad + (\psi_{11}\psi_{22} + \psi_{12}\psi_{21}) \text{Cov}[\varepsilon_1, \varepsilon_2]
\end{aligned}$$

- Set $\psi_{11} = \psi_{22} = 1$. This implies

$$\begin{aligned}
\text{Var}[y_1] &= \text{Var}[\varepsilon_1] + \psi_{12}^2 \text{Var}[\varepsilon_2] + 2\psi_{12} \text{Cov}[\varepsilon_1, \varepsilon_2] \\
\text{Var}[y_2] &= \psi_{21}^2 \text{Var}[\varepsilon_1] + \text{Var}[\varepsilon_2] + 2\psi_{21} \text{Cov}[\varepsilon_1, \varepsilon_2] \\
\text{Cov}[y_1, y_2] &= \psi_{21} \text{Var}[\varepsilon_1] + \psi_{12} \text{Var}[\varepsilon_2] \\
&\quad + (1 + \psi_{12}\psi_{21}) \text{Cov}[\varepsilon_1, \varepsilon_2]
\end{aligned}$$

- Still have three moments and five parameters

One normalization

- Equations are

$$\begin{aligned} y_{1,t} &= \psi_{11}\varepsilon_{1,t} + \psi_{12}\varepsilon_{2,t} \\ y_{2,t} &= \psi_{21}\varepsilon_{1,t} + \psi_{22}\varepsilon_{2,t} \end{aligned}$$

- One possible setup $\psi_{11} = \psi_{22} = 1$ and $\psi_{12} = \psi_{21} = 0$ in

$$\begin{aligned} \mathrm{Var}\left[y_1\right] &= \psi_{11}^2 \mathrm{Var}\left[\varepsilon_1\right] + \psi_{12}^2 \mathrm{Var}\left[\varepsilon_2\right] + 2\psi_{11}\psi_{12} \mathrm{Cov}\left[\varepsilon_1, \varepsilon_2\right] \\ \mathrm{Var}\left[y_2\right] &= \psi_{21}^2 \mathrm{Var}\left[\varepsilon_1\right] + \psi_{22}^2 \mathrm{Var}\left[\varepsilon_2\right] + 2\psi_{21}\psi_{22} \mathrm{Cov}\left[\varepsilon_1, \varepsilon_2\right] \\ \mathrm{Cov}\left[y_1, y_2\right] &= \psi_{11}\psi_{21} \mathrm{Var}\left[\varepsilon_1\right] + \psi_{12}\psi_{22} \mathrm{Var}\left[\varepsilon_2\right] \\ &\quad + \left(\psi_{11}\psi_{22} + \psi_{12}\psi_{21}\right) \mathrm{Cov}\left[\varepsilon_1, \varepsilon_2\right] \end{aligned}$$

which implies

$$\begin{aligned} \mathrm{Var}\left[y_1\right] &= \mathrm{Var}\left[\varepsilon_1\right] \\ \mathrm{Var}\left[y_2\right] &= \mathrm{Var}\left[\varepsilon_2\right] \\ \mathrm{Cov}\left[y_1, y_2\right] &= \mathrm{Cov}\left[\varepsilon_1, \varepsilon_2\right] \end{aligned}$$

A second normalization

- A second normalization
- Equations are

$$y_{1,t} = \psi_{11}\eta_{1,t} + \psi_{12}\eta_{2,t}$$
$$y_{2,t} = \psi_{21}\eta_{1,t} + \psi_{22}\eta_{2,t}$$

  - I use a different symbol for the innovations because these innovations may be different than the ones in the normalization above
- Another possible setup $\psi_{11} = \psi_{22} = 1$ and $\psi_{21} = 0$ and $\text{Cov}\left[\eta_1, \eta_2\right] = 0$
- This implies

$$y_{1,t} = \eta_{1,t} + \psi_{12}\eta_{2,t}$$
$$y_{2,t} = \eta_{2,t}$$

A second normalization

- The restrictions $\psi_{11} = \psi_{22} = 1$ and $\psi_{21} = 0$ and $\text{Cov}[\eta_1, \eta_2] = 0$ imply in

$$\begin{aligned} \text{Var}[y_1] &= \psi_{11}^2 \text{Var}[\eta_1] + \psi_{12}^2 \text{Var}[\eta_2] + 2\psi_{11}\psi_{12} \text{Cov}[\eta_1, \eta_2] \\ \text{Var}[y_2] &= \psi_{21}^2 \text{Var}[\eta_1] + \psi_{22}^2 \text{Var}[\eta_2] + 2\psi_{21}\psi_{22} \text{Cov}[\eta_1, \eta_2] \\ \text{Cov}[y_1, y_2] &= \psi_{11}\psi_{21} \text{Var}[\eta_1] + \psi_{12}\psi_{22} \text{Var}[\eta_2] \\ &\quad + (\psi_{11}\psi_{22} + \psi_{12}\psi_{21}) \text{Cov}[\eta_1, \eta_2] \end{aligned}$$

that

$$\begin{aligned} \text{Var}[y_1] &= \text{Var}[\eta_1] + \psi_{12}^2 \text{Var}[\eta_2] \\ \text{Var}[y_2] &= \text{Var}[\eta_2] \\ \text{Cov}[y_1, y_2] &= \psi_{12} \text{Var}[\eta_2] \end{aligned}$$

A second normalization

- In

$$\begin{aligned} \text{Var}[y_1] &= \text{Var}[\eta_1] + \psi_{12}^2 \text{Var}[\eta_2] \\ \text{Var}[y_2] &= \text{Var}[\eta_2] \\ \text{Cov}[y_1, y_2] &= \psi_{12} \text{Var}[\eta_2] \end{aligned}$$

- We can substitute $\text{Var}[\eta_2] = \text{Var}[y_2]$ into the other two equations to get

$$\begin{aligned} \text{Var}[\eta_1] &= \text{Var}[y_1] - \psi_{12}^2 \text{Var}[y_2] \\ \psi_{12} &= \frac{\text{Cov}[y_1, y_2]}{\text{Var}[y_2]} \end{aligned}$$

A second normalization

- In

$$\begin{aligned} \text{Var}[\eta_1] &= \text{Var}[y_1] - \psi_{12}^2 \text{Var}[y_2] \\ \psi_{12} &= \frac{\text{Cov}[y_1, y_2]}{\text{Var}[y_2]} \end{aligned}$$

- the coefficient ψ_{12} is just the regression coefficient in the regression of y_1 on y_2
- It might seem that $\text{Var}[\eta_1]$ could be negative, in which case this normalization is nonsense but

$$\begin{aligned} \text{Var}[\eta_1] &= \text{Var}[y_1] - \left(\frac{\text{Cov}[y_1, y_2]}{\text{Var}[y_2]}\right)^2 \text{Var}[y_2] \\ &= \text{Var}[y_1] - \frac{(\text{Cov}[y_1, y_2])^2}{\text{Var}[y_2]} \end{aligned}$$

A second normalization

- The variance is positive

$$\text{Var}\left[\eta_1\right] = \text{Var}\left[y_1\right] - \frac{\left(\text{Cov}\left[y_1, y_2\right]\right)^2}{\text{Var}\left[y_2\right]} > 0$$

if

$$\text{Var}\left[y_1\right] - \frac{\left(\text{Cov}\left[y_1, y_2\right]\right)^2}{\text{Var}\left[y_2\right]} > 0$$

$$\text{Var}\left[y_1\right] > \frac{\left(\text{Cov}\left[y_1, y_2\right]\right)^2}{\text{Var}\left[y_2\right]}$$

$$1 > \frac{\left(\text{Cov}\left[y_1, y_2\right]\right)^2}{\text{Var}\left[y_1\right]\text{Var}\left[y_2\right]}$$

- So the variance is positive as long as the squared correlation is less than unity

A second normalization

- In sum, the equations

$$y_{1,t} = \eta_{1,t} + \psi_{12}\eta_{2,t} \qquad \text{Cov}\left[\eta_{1,t}, \eta_{2,t}\right] = 0$$
$$y_{2,t} = \eta_{2,t}$$

- are related to the underlying moments of the data by

$$\text{Var}\left[\eta_1\right] = \text{Var}\left[y_1\right] - \psi_{12}^2 \, \text{Var}\left[y_2\right]$$
$$\text{Var}\left[\eta_2\right] = \text{Var}\left[y_2\right]$$
$$\psi_{12} = \frac{\text{Cov}\left[y_1, y_2\right]}{\text{Var}\left[y_2\right]}$$

Comments on the two normalizations

- The innovations in the representation

$$y_{1,t} = \varepsilon_{1,t}$$
$$y_{2,t} = \varepsilon_{2,t}$$
$$\text{Cov}\left[\varepsilon_1, \varepsilon_2\right] = \text{Cov}\left[y_1, y_1\right]$$

are different than the innovations in the representation

$$y_{1,t} = \eta_{1,t} + \psi_{12}\eta_{2,t}$$
$$y_{2,t} = \eta_{2,t}$$
$$\text{Cov}\left[\eta_1, \eta_2\right] = 0$$

- They are different specializations of the same general set of equations

$$y_{1,t} = \psi_{11}\varepsilon_{1,t} + \psi_{12}\varepsilon_{2,t}$$
$$y_{2,t} = \psi_{21}\varepsilon_{1,t} + \psi_{22}\varepsilon_{2,t}$$

Comments on the two normalizations

- The two normalizations are representations of the same data because

$$\text{Var}\left[\varepsilon_1\right] = \text{Var}\left[y_1\right]$$
$$\text{Var}\left[\varepsilon_2\right] = \text{Var}\left[y_2\right]$$
$$\text{Cov}\left[\varepsilon_1, \varepsilon_2\right] = \text{Cov}\left[y_1, y_2\right]$$

- and

$$\text{Var}\left[\eta_1\right] = \text{Var}\left[y_1\right] - \psi_{12}^2 \text{Var}\left[y_2\right]$$
$$\text{Var}\left[\eta_2\right] = \text{Var}\left[y_2\right]$$
$$\psi_{12} = \frac{\text{Cov}\left[y_1, y_2\right]}{\text{Var}\left[y_2\right]}$$

with

$$\text{Cov}\left[\eta_1, \eta_2\right] = 0$$

A third normalization

- Another normalization of

$$y_{1,t} = \psi_{11}\varepsilon_{1,t} + \psi_{12}\varepsilon_{2,t}$$
$$y_{2,t} = \psi_{21}\varepsilon_{1,t} + \psi_{22}\varepsilon_{2,t}$$

  is $\psi_{11} = \psi_{22} = 1$ and $\psi_{12} = 0$ and $\text{Cov}[\varepsilon_1, \varepsilon_2] = 0$ which implies

$$y_{1,t} = \nu_{1,t}$$
$$y_{2,t} = \psi_{21}\nu_{1,t} + \nu_{2,t}$$
$$\text{Cov}[\nu_1, \nu_2] = 0$$

- Do this yourself

Three different normalizations fit the data equally well

- The general equation has unidentified parameters

$$y_{1,t} = \psi_{11}\varepsilon_{1,t} + \psi_{12}\varepsilon_{2,t}$$
$$y_{2,t} = \psi_{21}\varepsilon_{1,t} + \psi_{22}\varepsilon_{2,t}$$

with unrestricted $\text{Var}\left[\varepsilon_1\right]$, $\text{Var}\left[\varepsilon_2\right]$ and $\text{Cov}\left[\varepsilon_1, \varepsilon_2\right]$

Three different normalizations can be computed and fit the data equally well

- One normalization

$$y_{1,t} = \varepsilon_{1,t}$$
$$y_{2,t} = \varepsilon_{2,t}$$
$$\text{Cov}\left[\varepsilon_1, \varepsilon_2\right] = \text{Cov}\left[y_1, y_1\right]$$

- A second normalization

$$y_{1,t} = \eta_{1,t} + \psi_{12}\eta_{2,t}$$
$$y_{2,t} = \eta_{2,t}$$
$$\text{Cov}\left[\eta_1, \eta_2\right] = 0$$

- A third normalization

$$y_{1,t} = \nu_{1,t}$$
$$y_{2,t} = \psi_{21}\nu_{1,t} + \nu_{2,t}$$
$$\text{Cov}\left[\nu_1, \nu_2\right] = 0$$

A different way of looking at this issue

- The data are serially uncorrelated and it might seem most natural just to use

$$y_{1,t} = \varepsilon_{1,t}$$
$$y_{2,t} = \varepsilon_{2,t}$$
$$\text{Cov}\left[\varepsilon_1, \varepsilon_2\right] = \text{Cov}\left[y_1, y_1\right]$$

- But what if $y_{2,t}$ helps to determine $y_{1,t}$ but not vice versa?

$$y_{1,t} = \varepsilon_{1,t} + \psi_{12} y_{2,t}$$
$$y_{2,t} = \varepsilon_{2,t}$$

  - Estimation by ordinary least squares implies $\text{Cov}\left[\varepsilon_1, \varepsilon_2\right] = 0$
- Important to note there is the third normalization with $y_{2,t} = \psi_{21} y_{1,t} + \varepsilon_{2,t}$ which fits the data equally well

Three different normalizations can be computed and fit the data equally well

- Only economic theory – if anything – can tell us which is the most informative way of looking at the data
  - All of them are **observationally equivalent** in the sense that they fit the data equally well
  - Equally well: the values of the likelihood functions are identical
  - The data will not tell us which of these three – and probably other representations – is inconsistent with the data
  - All three representations are consistent with the data
- There is no sense of the word in which we can infer from the data that one variable determines another or is exogenous to the other
- This problem of reducing the set of parameters to ones which are interesting and estimable goes under the name of **identification**

Normalization in vector autoregressions

- The same issue arises in vector autoregressions

$$\mathbf{y}_t = \mathbf{A}_0 + \mathbf{A}_1 (\mathrm{L})\, \mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t$$

  - Suppress the constant term to simply the algebra
  - Look at only two variables
  - Only one lag

$$\begin{aligned} y_{1,t} &= a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \varepsilon_{1,t} \\ y_{2,t} &= a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \varepsilon_{2,t} \end{aligned}$$

  - No obvious reason why correlation of $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ need be zero
  - And the correlation won't be zero in general

Ordering of vector autoregressions

- Instead of

$$
\begin{aligned}
y_{1,t} &= a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \varepsilon_{1,t} \\
y_{2,t} &= a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \varepsilon_{2,t}
\end{aligned}
$$

- Estimate

$$
\begin{aligned}
y_{1,t} &= a_1^* y_{2,t} + a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \eta_{1,t} \\
y_{2,t} &= a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \eta_{2,t}
\end{aligned}
$$

- Current value of $y_{2,t}$ appears in the equation for $y_{1,t}$
- What is correlation of $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$?
- What is correlation of $\eta_{1,t}$ and $\eta_{2,t}$?

Correlation of error terms with one variable in the other equation

- Why is the correlation of error terms zero in

$$y_{1,t} = a_1^* y_{2,t} + a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \eta_{1,t}$$
$$y_{2,t} = a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \eta_{2,t}$$

Correlation of error terms with one variable in the other equation

- Why is the correlation of error terms zero in

$$y_{1,t} = a_1^* y_{2,t} + a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \eta_{1,t}$$
$$y_{2,t} = a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \eta_{2,t}$$

- Called "regressions" $\Leftrightarrow$ estimated by least squares

Correlation of error terms with one variable in the other equation

- Why is the correlation of error terms zero in

$$
\begin{aligned}
y_{1,t} &= a_1^* y_{2,t} + a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \eta_{1,t} \\
y_{2,t} &= a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \eta_{2,t}
\end{aligned}
$$

- Called "regressions" $\Leftrightarrow$ estimated by least squares
- Covariance of $\eta_{1,t}$ and $\eta_{2,t}$ is given by $\mathrm{E}\left[\eta_{1,t}\eta_{2,t}\right]$

Correlation of error terms with one variable in the other equation

- Why is the correlation of error terms zero in

$$y_{1,t} = a_1^* y_{2,t} + a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \eta_{1,t}$$
$$y_{2,t} = a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \eta_{2,t}$$

- Called "regressions" $\Leftrightarrow$ estimated by least squares
- Covariance of $\eta_{1,t}$ and $\eta_{2,t}$ is given by $\mathrm{E}\left[\eta_{1,t}\eta_{2,t}\right]$
- Know $\mathrm{E}\left[\eta_{1,t} y_{2,t}\right] = 0$ by least squares orthogonality

Correlation of error terms with one variable in the other equation

- Why is the correlation of error terms zero in

$$
\begin{aligned}
y_{1,t} &= a_1^* y_{2,t} + a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \eta_{1,t} \\
y_{2,t} &= a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \eta_{2,t}
\end{aligned}
$$

- Called "regressions" $\Leftrightarrow$ estimated by least squares
- Covariance of $\eta_{1,t}$ and $\eta_{2,t}$ is given by $\mathrm{E}\left[\eta_{1,t}\eta_{2,t}\right]$
- Know $\mathrm{E}\left[\eta_{1,t} y_{2,t}\right] = 0$ by least squares orthogonality
- Also know

$$
\begin{aligned}
\mathrm{E}\left[\eta_{1,t} y_{2,t}\right] &= \mathrm{E}\left[\eta_{1,t}\left(a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \eta_{2,t}\right)\right] \\
&= a_{2,1}\,\mathrm{E}\,\eta_{1,t} y_{1,t-1} + a_{2,2}\,\mathrm{E}\,\eta_{1,t} y_{2,t-1} + \mathrm{E}\left[\eta_{1,t}\eta_{2,t}\right] \\
&= \mathrm{E}\left[\eta_{1,t}\eta_{2,t}\right]
\end{aligned}
$$

Correlation of error terms with one variable in the other equation

- In

$$y_{1,t} = a_1^* y_{2,t} + a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \eta_{1,t}$$
$$y_{2,t} = a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \eta_{2,t}$$

- Know

$$\mathrm{E}\left[\eta_{1,t} y_{2,t}\right] = 0$$

- Also know

$$\mathrm{E}\left[\eta_{1,t} y_{2,t}\right] = \mathrm{E}\left[\eta_{1,t} \eta_{2,t}\right]$$

- Therefore

$$\mathrm{E}\left[\eta_{1,t} \eta_{2,t}\right] = 0$$

Correlation of error terms with one variable in the other equation

- What if we estimated instead

$$y_{1,t} = a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \nu_{1,t}$$
$$y_{2,t} = a_2^* y_{1,t} + a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \nu_{2,t}$$

- Then

$$\mathrm{E}\left[\nu_{1,t}\nu_{2,t}\right] = 0$$

- Do yourself

Vector autoregressive representations

- All three of the systems

$$
\begin{aligned}
y_{1,t} =& a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \varepsilon_{1,t} \\
y_{2,t} =& a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \varepsilon_{2,t}
\end{aligned}
$$

- and

$$
\begin{aligned}
y_{1,t} =& a_1^* y_{2,t} + a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \eta_{1,t} \\
y_{2,t} =& a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \eta_{2,t}
\end{aligned}
$$

- and

$$
\begin{aligned}
y_{1,t} =& a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \nu_{1,t} \\
y_{2,t} =& a_2^* y_{1,t} + a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \nu_{2,t}
\end{aligned}
$$

  fit the data equally well
- The correlation of $y_{1,t}$ and $y_{2,t}$ over and above dependence on lagged values just appears in different places

Vector autoregressive representations

- In

$$\begin{aligned} y_{1,t} &= a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \varepsilon_{1,t} \\ y_{2,t} &= a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \varepsilon_{2,t} \end{aligned}$$

  correlation of current unpredictable changes appears in correlation of error term

- In

$$\begin{aligned} y_{1,t} &= a_1^* y_{2,t} + a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \eta_{1,t} \\ y_{2,t} &= a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \eta_{2,t} \end{aligned}$$

  correlation of current unpredictable changes appears in coefficient a_1^*

- In

$$\begin{aligned} y_{1,t} &= a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \nu_{1,t} \\ y_{2,t} &= a_2^* y_{1,t} + a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \nu_{2,t} \end{aligned}$$

  correlation of current unpredictable changes appears in coefficient a_2^*

Government expenditures and real GDP

- Does this matter?
- Government expenditures and real GDP from Andrew Young "Why in the world are we all Keynesians now?", page 12
  - Graphs show response of real GDP to a one-standard deviation "shock" to government spending
  - Top graph has current government expenditures appearing in the real GDP equation

$$y_{1,t} = a_1^* y_{2,t} + a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \eta_{1,t}$$
$$y_{2,t} = a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \eta_{2,t}$$

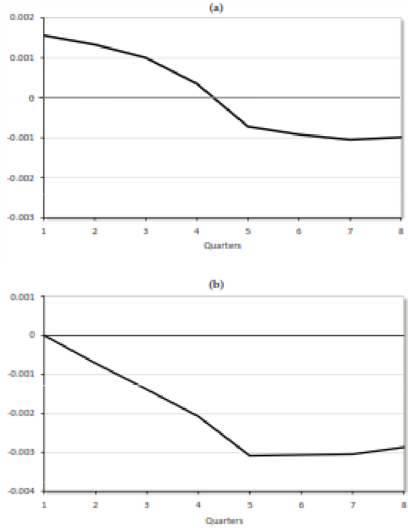
    - ★ Effect of one-standard-deviation change of $\eta_{2,t}$ on $y_{1,t}, y_{1,t+1}, \ldots$
  - Bottom graph has current GDP appearing in the current government expenditures equation

$$y_{1,t} = a_{1,1} y_{1,t-1} + a_{1,2} y_{2,t-1} + \nu_{1,t}$$
$$y_{2,t} = a_2^* y_{1,t} + a_{2,1} y_{1,t-1} + a_{2,2} y_{2,t-1} + \nu_{2,t}$$

    - ★ Effect of one-standard-deviation change of $\nu_{2,t}$ on $y_{1,t}, y_{1,t+1}, \ldots$

Government expenditures and real GDP

Figure 4
Estimated Effects of Real GDP to a Federal Government Expenditures Shock



Why not estimate simple VAR?

- Why not use

$$y_{1,t} = a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \varepsilon_{1,t}$$
$$y_{2,t} = a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \varepsilon_{2,t}$$

Why not estimate simple VAR?

- Why not use

$$y_{1,t} = a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \varepsilon_{1,t}$$
$$y_{2,t} = a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \varepsilon_{2,t}$$

- Effect of one-standard-deviation change of $\varepsilon_{2,t}$ on $y_{1,t}$ doesn't represent change in government spending alone if the correlation of $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ is not zero

Why not estimate simple VAR?

- Why not use

$$y_{1,t} = a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \varepsilon_{1,t}$$
$$y_{2,t} = a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \varepsilon_{2,t}$$

- Effect of one-standard-deviation change of $\varepsilon_{2,t}$ on $y_{1,t}$ doesn't represent change in government spending alone if the correlation of $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ is not zero
- There is a correlated change in $y_{2,t}$

Why not estimate simple VAR?

- Why not use

$$y_{1,t} = a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \varepsilon_{1,t}$$
$$y_{2,t} = a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \varepsilon_{2,t}$$

- Effect of one-standard-deviation change of $\varepsilon_{2,t}$ on $y_{1,t}$ doesn't represent change in government spending alone if the correlation of $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ is not zero
- There is a correlated change in $y_{2,t}$
- If the correlation of $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ is zero *in this representation*, then can just use this representation

Why not estimate simple VAR?

- Why not use

$$y_{1,t} = a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \varepsilon_{1,t}$$
$$y_{2,t} = a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \varepsilon_{2,t}$$

- Effect of one-standard-deviation change of $\varepsilon_{2,t}$ on $y_{1,t}$ doesn't represent change in government spending alone if the correlation of $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ is not zero
- There is a correlated change in $y_{2,t}$
- If the correlation of $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ is zero *in this representation*, then can just use this representation
  - Why can one use this?

Why not estimate simple VAR?

- Why not use

$$y_{1,t} = a_{1,1}y_{1,t-1} + a_{1,2}y_{2,t-1} + \varepsilon_{1,t}$$
$$y_{2,t} = a_{2,1}y_{1,t-1} + a_{2,2}y_{2,t-1} + \varepsilon_{2,t}$$

- Effect of one-standard-deviation change of $\varepsilon_{2,t}$ on $y_{1,t}$ doesn't represent change in government spending alone if the correlation of $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ is not zero
- There is a correlated change in $y_{2,t}$
- If the correlation of $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ is zero *in this representation*, then can just use this representation
    - Why can one use this?
    - What would be estimated coefficient if one estimated an equation in which the current value of $y_{2,t}$ appeared in the $y_{1,t}$ equation?

In short

- Identification in this context is choosing a normalization of the VAR
  - Cannot be determined based on the data
  - Makes it possible to estimate parameters in some specification
  - Can have important effects on empirical estimates

Summary I

- A vector autoregression (VAR) is a very general representation of a time series
- Any set of covariance stationary time series has a VAR representation with serially uncorrelated innovations and potentially an infinite number of lags
- Normalizations are necessary to obtain a representation for a VAR
  - Problem is very obvious for moving average
  - There for VAR as well
- There is a normalization to identify a VAR
- Issue is how to identify the relationship between contemporaneous values of the variables
- This relationship can appear in
  - Correlation of innovations
  - Coefficient of a variable in the other equation or equations
- Statistical criteria cannot determine the choice of which current values of variables appear in which equations

Summary II

- More generally, Ordinary Least Squares may not even be the right way to summarize the relationship for some purposes