Difference Equations

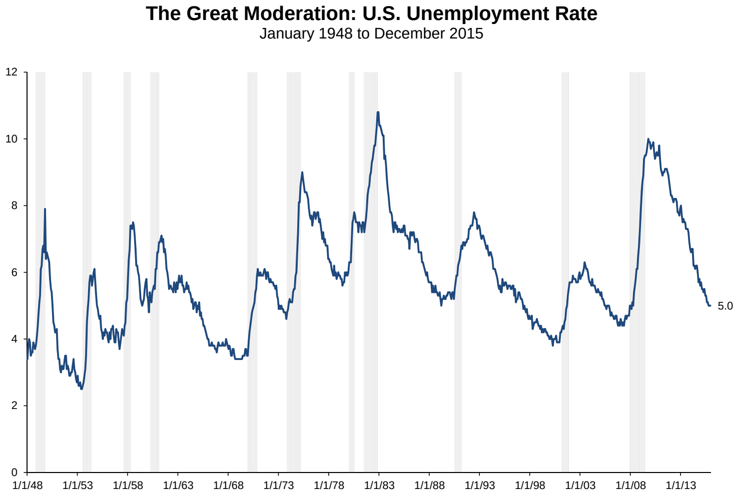
Gerald P. Dwyer

Clemson University

January 2016

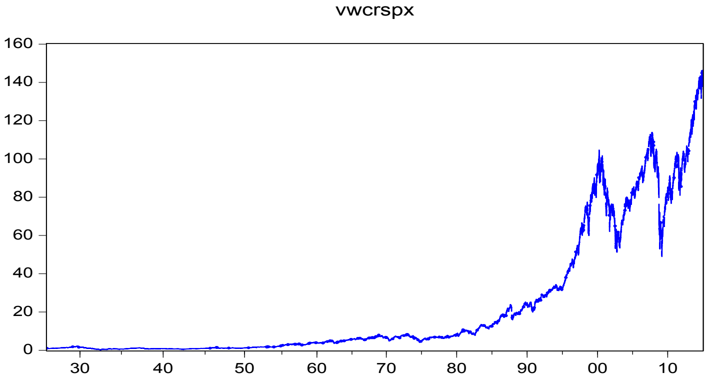
Outline

Times Series – Unemployment rate

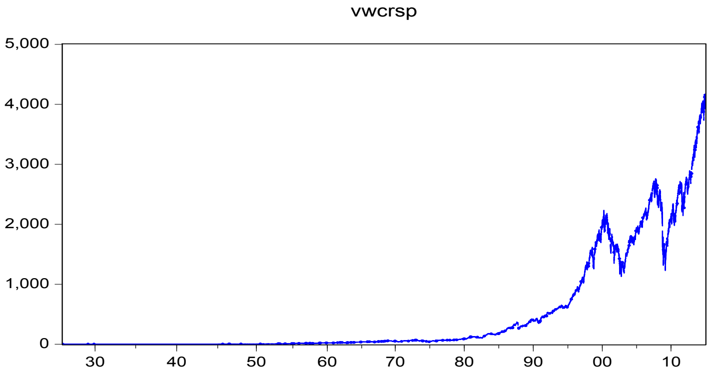


Sources: BLS, NBER, Haver Analytics

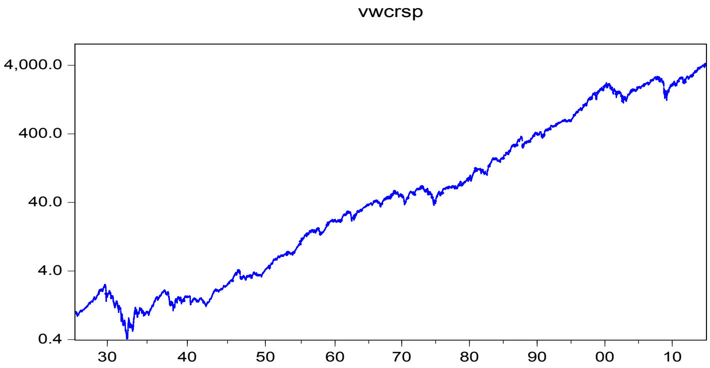
Times Series – CRSP stock price index



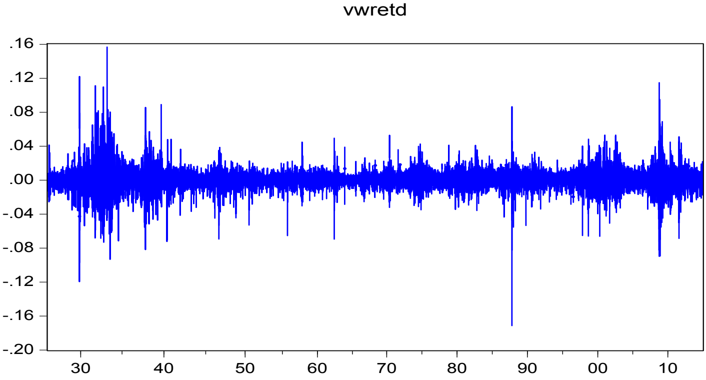
Times Series – CRSP stock price index



Times Series – CRSP stock price index



Times Series – Return on CRSP stock index



Autoregressions

- Unemployment rate
- CRSP price index
- CRSP return

Terminology I

- Time-series analysis makes heavy use of terms such as "stable" and "roots of the equation", even "eigenvalues" and "eigenvectors"
- What does all this mean?
- A first-order linear difference equation (one lagged value of variable on left)

$$y_t = a_0 + a_1 y_{t-1} + x_t, \;\; t = 1, ..., T$$

  - The variable y at period t has the value represented by y_t
  - The variable y_{t-1} is the same variable one period earlier
  - Time, t, runs from 1 to T
  - a_0 and a_1 are constant coefficients in the equation
  - x_t is a forcing process that affects y_t
    - x_t can be deterministic or stochastic

Terms "First Order" and "Linear"

- Why is this equation called a first-order linear difference equation?

$$y_t = a_0 + a_1 y_{t-1} + x_t, \;\; t = 1, ..., T$$

  - Difference – can write in terms of first difference

$$y_t - y_{t-1} = a_0 + (a_1 - 1)\, y_{t-1} + x_t, \;\; t = 1, ..., T$$

  - Linear – involves no nonlinear functions of y_t

Order of Difference Equation

- The "order" of a difference equation is the maximum lag included on the right-hand side
- First-order difference equation

$$y_t = a_0 + a_1 y_{t-1} + x_t$$

- Second-order difference equation

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + x_t$$

- k'th order difference equation

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \ldots + a_k y_{t-k} + x_t$$

- Another k'th order difference equation

$$y_t = a_0 + a_k y_{t-k} + x_t$$

- Can also define in terms of number of differences can take of y_t but that is not particularly helpful and it's the same

Deterministic and stochastic

- "Deterministic" means perfectly predictable from its own past
  - For example trend $x_t = bt$ with b known
  - Quarterly dummy variables
  - A deterministic difference equation

$$y_t = a_0 + a_1 y_{t-1} + bt, \;\; t = 1, ..., T$$

- "Stochastic" means evolves according to a probability law, e.g. $x_t = \varepsilon_t \sim N(0, 1)$
  - Another word is "random"
    - ★ Does not mean "arbitrary"
  - "Random" means for example $x_t = \varepsilon_t \sim N(0, 1)$
  - A stochastic difference equation

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t, \;\; t = 1, ..., T$$

$$\varepsilon_t \sim N\left(0, \sigma^2\right)$$

Behavior of difference equations I

- First-order linear difference equation with no forcing variable

$$y_t = a_0 + a_1 y_{t-1}, \;\; t = 1, \ldots, T$$

- At time 1, this is

$$y_1 = a_0 + a_1 y_0$$

- At time 2, it is

$$y_2 = a_0 + a_1 y_1$$

  which also equals

$$y_2 = a_0 + a_1 \left(a_0 + a_1 y_0 \right) = a_0 + a_1 a_0 + a_1^2 y_0$$

- At time 3,

$$y_3 = a_0 + a_1 \left(a_0 + a_1 a_0 + a_1^2 y_0 \right) = a_0 + a_1 a_0 + a_1^2 a_0 + a_1^3 y_0$$

Behavior of difference equations II

- At time 4,

$$\begin{aligned} y_4 &= a_0 + a_1 \left(a_0 + a_1 a_0 + a_1^2 a_0 + a_1^3 y_0 \right) \\ &= a_0 + a_1 a_0 + a_1^2 a_0 + a_1^3 a_0 + a_1^4 y_0 \end{aligned}$$

- Can prove by induction

$$y_t = \sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0, \;\; t = 1, ..., T$$

- Well defined paths depending on value of a_1 – see spreadsheet

Operators I

- Two operators are often used in analyzing difference equations
  - An operator is a symbol that indicates an algebraic operation to be taken
  - A plus sign $(+)$ is an example of an operator, as are minus $(-)$, times $(\cdot$ and x), and division $(/)$
- Difference operator Δ

$$
\begin{aligned}
\Delta y_t &= y_t - y_{t-1} \\
\Delta y_{t+h} &= y_{t+h} - y_{t+h-1}, \; h = ..., -1, 0.1, \ldots \\
\Delta^2 y_t &= \Delta \Delta y_t = \Delta \left(y_t - y_{t-1} \right) = \Delta y_t - \Delta y_{t-1} \\
&= y_t - 2y_{t-1} + y_{t-2}
\end{aligned}
$$

  - Higher orders of differencing possible too
  - Statisticians sometimes use ∇

Operators II

- Lag operator L

$$\mathrm{L}\, y_t = y_{t-1}$$
$$\mathrm{L}^2\, y_t = y_{t-2}$$
$$\mathrm{L}^i\, y_t = y_{t-i}$$

  - Statisticians sometimes use B (backshift)

Polynomial in lag operator

- When more than one lag, often form polynomial in lag operator
- $y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + x_t$

Polynomial in lag operator

- When more than one lag, often form polynomial in lag operator
- $y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + x_t$
- $y_t = a_0 + a_1 \, \mathrm{L} \, y_t + a_2 \, \mathrm{L}^2 \, y_t + x_t$

Polynomial in lag operator

- When more than one lag, often form polynomial in lag operator
- $y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + x_t$
- $y_t = a_0 + a_1 \, \mathrm{L} \, y_t + a_2 \, \mathrm{L}^2 \, y_t + x_t$
- $y_t - a_1 \, \mathrm{L} \, y_t - a_2 \, \mathrm{L}^2 \, y_t = a_0 + x_t$

Polynomial in lag operator

- When more than one lag, often form polynomial in lag operator
- $y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + x_t$
- $y_t = a_0 + a_1 \mathrm{L}\, y_t + a_2 \mathrm{L}^2\, y_t + x_t$
- $y_t - a_1 \mathrm{L}\, y_t - a_2 \mathrm{L}^2\, y_t = a_0 + x_t$
- $\left(1 - a_1 \mathrm{L} - a_2 \mathrm{L}^2\right) y_t = a_0 + x_t$

Polynomial in lag operator

- When more than one lag, often form polynomial in lag operator
- $y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + x_t$
- $y_t = a_0 + a_1 \mathrm{L}\, y_t + a_2 \mathrm{L}^2\, y_t + x_t$
- $y_t - a_1 \mathrm{L}\, y_t - a_2 \mathrm{L}^2\, y_t = a_0 + x_t$
- $\left(1 - a_1 \mathrm{L} - a_2 \mathrm{L}^2\right) y_t = a_0 + x_t$
- $a(\mathrm{L})\, y_t = a_0 + x_t,\ a(\mathrm{L}) = 1 - a_1 \mathrm{L} - a_2 \mathrm{L}^2$

Polynomial in lag operator

- When more than one lag, often form polynomial in lag operator
- $y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + x_t$
- $y_t = a_0 + a_1 \mathrm{L}\, y_t + a_2 \mathrm{L}^2\, y_t + x_t$
- $y_t - a_1 \mathrm{L}\, y_t - a_2 \mathrm{L}^2\, y_t = a_0 + x_t$
- $\left(1 - a_1 \mathrm{L} - a_2 \mathrm{L}^2\right) y_t = a_0 + x_t$
- $a(\mathrm{L})\, y_t = a_0 + x_t,\ a(\mathrm{L}) = 1 - a_1 \mathrm{L} - a_2 \mathrm{L}^2$
- or $y_t = a_0 + a^*(\mathrm{L})\, y_{t-1} + x_t,\ a^*(\mathrm{L}) = \sum_{i=1}^{1} a_i \mathrm{L}^i = a_1 + a_2 \mathrm{L}$

Lag Operator

- Lag operator rules
- L^{-i} is well defined

$$\mathrm{L}^{-i}\, y_t = y_{t+i}$$

- Lag operator applied to a constant c

$$\mathrm{L}\, c = c$$

- Distributive

$$\left(\mathrm{L}^i + \mathrm{L}^j\right) y_t = y_{t-i} + y_{t-j}$$

- Associative

$$\mathrm{L}^i\, \mathrm{L}^j\, y_t = \mathrm{L}^j\, \mathrm{L}^i\, y_t = \mathrm{L}^{i+j}\, y_t = \mathrm{L}^{j+i}\, y_t = y_{t-(i+j)} = y_{t-i-j}$$

- Same rules for the difference operator Δ
- Note $\Delta y_t = (1 - \mathrm{L})\, y_t$

Multipliers and Impulse response functions

- The multiplier in period t for any period $t + h$ indicates the response of a variable h periods in the future to an impulse in some period t
  - First-order difference equation

$$y_t = a_0 + a_1 y_{t-1} + x_t, \;\; t = 1, T \tag{1}$$

  - The h'th period multiplier simply is

$$\frac{\partial y_{t+h}}{\partial x_t}$$

    - Holding constant any other forcing variables

Multipliers and Impulse response functions

- The multiplier in period t for any period $t+h$ indicates the response of a variable h periods in the future to an impulse in some period t
  - First-order difference equation

$$y_t = a_0 + a_1 y_{t-1} + x_t, \;\; t = 1, T \tag{1}$$

  - The h'th period multiplier simply is

$$\frac{\partial y_{t+h}}{\partial x_t}$$

    - Holding constant any other forcing variables
  - If you iterate equation (1) with one initial nonzero forcing value x_t, you will find

$$y_{t+h} = \sum_{i=0}^{h} a_1^i a_0 + a_1^{h+1} y_{t-1} + a_1^h x_t$$

  - Therefore

$$\frac{\partial y_{t+h}}{\partial x_t} = a_1^h$$

    for a first-order difference equation

Impulse response function

- The impulse response function is the set of

$$\left\{ \frac{\partial y_{t+h}}{\partial x_t} \right\}, \quad n = 0, 1, 2, \ldots$$

  - For a linear first-order difference equation, this is just

$$\left\{ \frac{\partial y_{t+h}}{\partial x_t} = a_1^h \right\}, \quad h = 0, 1, 2, \ldots$$

- Properties for a linear equation such as this one
  - The size of the impulse response is independent of the magnitude of x_t or y_t
  - The size of the impulse response is independent of the particular time period t
    - ★ All that matters is a_1 and h

Solution

- A "solution" to a difference equation is a representation of the difference equation in which the value depends on the coefficients, the sequence of x_t values, denoted $\{x_t\}$ and possibly one or more initial values
- For example, for the first-order difference equation with no forcing variable

$$y_t = a_0 + a_1 y_{t-1} \tag{2}$$

  - a solution is the function

$$y_t = \sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0, \;\; t = 1, ..., T$$

  - which we determined above

Verify solution

- It is possible to verify this is a solution by substituting it into the left and right-hand sides of equation (2)
- $y_t = a_0 + a_1 y_{t-1}$ with solution $y_t = \sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0$

Verify solution

- It is possible to verify this is a solution by substituting it into the left and right-hand sides of equation (2)
- $y_t = a_0 + a_1 y_{t-1}$ with solution $y_t = \sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0$
- $\sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0 = a_0 + a_1 \left(\sum_{i=0}^{t-2} a_1^i a_0 + a_1^{t-1} y_0 \right)$

Verify solution

- It is possible to verify this is a solution by substituting it into the left and right-hand sides of equation (2)
- $y_t = a_0 + a_1 y_{t-1}$ with solution $y_t = \sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0$
- $\sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0 = a_0 + a_1 \left(\sum_{i=0}^{t-2} a_1^i a_0 + a_1^{t-1} y_0 \right)$
- $\sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0 = a_0 + \left(\sum_{i=1}^{t-1} a_1^i a_0 + a_1^t y_0 \right)$

Verify solution

- It is possible to verify this is a solution by substituting it into the left and right-hand sides of equation (2)
- $y_t = a_0 + a_1 y_{t-1}$ with solution $y_t = \sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0$
- $\sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0 = a_0 + a_1 \left(\sum_{i=0}^{t-2} a_1^i a_0 + a_1^{t-1} y_0 \right)$
- $\sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0 = a_0 + \left(\sum_{i=1}^{t-1} a_1^i a_0 + a_1^t y_0 \right)$
- $\sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0 = \sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0$

Verify solution

- It is possible to verify this is a solution by substituting it into the left and right-hand sides of equation (2)
- $y_t = a_0 + a_1 y_{t-1}$ with solution $y_t = \sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0$
- $\sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0 = a_0 + a_1 \left(\sum_{i=0}^{t-2} a_1^i a_0 + a_1^{t-1} y_0 \right)$
- $\sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0 = a_0 + \left(\sum_{i=1}^{t-1} a_1^i a_0 + a_1^t y_0 \right)$
- $\sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0 = \sum_{i=0}^{t-1} a_1^i a_0 + a_1^t y_0$
- Therefore holds as an identity

Methods to solve difference equations

1. Iteration
2. General method
3. Method of undetermined coefficients

Another example of solution by iteration

- Equation is

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t, \ \varepsilon_t \sim N(0, \sigma^2)$$

  - No initial condition – push back to minus infinity instead of having initial condition

- Solve iteratively backward

- $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$

Another example of solution by iteration

- Equation is

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t,\ \ \varepsilon_t \sim N(0, \sigma^2)$$

  - No initial condition – push back to minus infinity instead of having initial condition

- Solve iteratively backward

- $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$
- $y_{t-1} = a_0 + a_1 y_{t-2} + \varepsilon_{t-1}$

Another example of solution by iteration

- Equation is
$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t,\ \ \varepsilon_t \sim N(0, \sigma^2)$$
  - No initial condition – push back to minus infinity instead of having initial condition
- Solve iteratively backward
- $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$
- $y_{t-1} = a_0 + a_1 y_{t-2} + \varepsilon_{t-1}$
- Therefore $y_t = a_0 + a_1 \left(a_0 + a_1 y_{t-2} + \varepsilon_{t-1} \right) + \varepsilon_t =$ $a_0 + a_1 a_0 + a_1^2 y_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$

Another example of solution by iteration

- Equation is
$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t, \ \varepsilon_t \sim N(0, \sigma^2)$$
  - No initial condition – push back to minus infinity instead of having initial condition
- Solve iteratively backward

- $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$
- $y_{t-1} = a_0 + a_1 y_{t-2} + \varepsilon_{t-1}$
- Therefore $y_t = a_0 + a_1 \left(a_0 + a_1 y_{t-2} + \varepsilon_{t-1}\right) + \varepsilon_t =$ $a_0 + a_1 a_0 + a_1^2 y_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$
- $y_t = a_0 + a_1 a_0 + a_1^2 a_0 + a_1^3 y_{t-3} + a_1^2 \varepsilon_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$

Another example of solution by iteration

- Equation is
$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t, \ \varepsilon_t \sim N(0, \sigma^2)$$
  - No initial condition – push back to minus infinity instead of having initial condition
- Solve iteratively backward
- $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$
- $y_{t-1} = a_0 + a_1 y_{t-2} + \varepsilon_{t-1}$
- Therefore $y_t = a_0 + a_1 \left(a_0 + a_1 y_{t-2} + \varepsilon_{t-1}\right) + \varepsilon_t = a_0 + a_1 a_0 + a_1^2 y_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$
- $y_t = a_0 + a_1 a_0 + a_1^2 a_0 + a_1^3 y_{t-3} + a_1^2 \varepsilon_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$
- $y_t = \sum_{i=0}^{t-1} a_0 a_1^i + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_1^t y_0$

Another example of solution by iteration

- Equation is
$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t, \;\; \varepsilon_t \sim N(0, \sigma^2)$$
  - No initial condition – push back to minus infinity instead of having initial condition
- Solve iteratively backward
- $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$
- $y_{t-1} = a_0 + a_1 y_{t-2} + \varepsilon_{t-1}$
- Therefore $y_t = a_0 + a_1 \left(a_0 + a_1 y_{t-2} + \varepsilon_{t-1} \right) + \varepsilon_t = a_0 + a_1 a_0 + a_1^2 y_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$
- $y_t = a_0 + a_1 a_0 + a_1^2 a_0 + a_1^3 y_{t-3} + a_1^2 \varepsilon_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$
- $y_t = \sum_{i=0}^{t-1} a_0 a_1^i + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_1^t y_0$
- If $|a_1| < 1$, then perhaps $y_t = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$

Another example of solution by iteration

- Equation is
$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t, \ \varepsilon_t \sim N(0, \sigma^2)$$
  - No initial condition – push back to minus infinity instead of having initial condition
- Solve iteratively backward
- $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$
- $y_{t-1} = a_0 + a_1 y_{t-2} + \varepsilon_{t-1}$
- Therefore $y_t = a_0 + a_1 \left(a_0 + a_1 y_{t-2} + \varepsilon_{t-1}\right) + \varepsilon_t = a_0 + a_1 a_0 + a_1^2 y_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$
- $y_t = a_0 + a_1 a_0 + a_1^2 a_0 + a_1^3 y_{t-3} + a_1^2 \varepsilon_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$
- $y_t = \sum_{i=0}^{t-1} a_0 a_1^i + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_1^t y_0$
- If $|a_1| < 1$, then perhaps $y_t = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$
- Cannot solve this way if $a_1 = 1$ or $|a_1| > 1$

Solution?

- If $|a_1| < 1$, then $y_t = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$ is a solution

Solution?

- Have $y_t = \sum_{i=0}^{t-1} a_0 a_1^i + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_1^t y_0$
- Want limit as push time 0 further back
- Use t_0 instead of 0 for initial time period which implies $y_t = \sum_{i=0}^{t-1-t_0} a_0 a_1^i + \sum_{i=0}^{t-1-t_0} a_1^i \varepsilon_{t-i} + a_1^{t-t_0} y_{t_0}$

Solution?

- Have $y_t = \sum_{i=0}^{t-1} a_0 a_1^i + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_1^t y_0$
- Want limit as push time 0 further back
- Use t_0 instead of 0 for initial time period which implies
  $y_t = \sum_{i=0}^{t-1-t_0} a_0 a_1^i + \sum_{i=0}^{t-1-t_0} a_1^i \varepsilon_{t-i} + a_1^{t-t_0} y_{t_0}$
- Let $h = t - t_0$ which implies $y_t = \sum_{i=0}^{h-1} a_0 a_1^i + \sum_{i=0}^{h-1} a_1^i \varepsilon_{t-i} + a_1^h y_{t-h}$

Solution?

- Have $y_t = \sum_{i=0}^{t-1} a_0 a_1^i + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_1^t y_0$
- Want limit as push time 0 further back
- Use t_0 instead of 0 for initial time period which implies $y_t = \sum_{i=0}^{t-1-t_0} a_0 a_1^i + \sum_{i=0}^{t-1-t_0} a_1^i \varepsilon_{t-i} + a_1^{t-t_0} y_{t_0}$
- Let $h = t - t_0$ which implies $y_t = \sum_{i=0}^{h-1} a_0 a_1^i + \sum_{i=0}^{h-1} a_1^i \varepsilon_{t-i} + a_1^h y_{t-h}$
- $\lim_{h\to\infty} \sum_{i=0}^{h-1} a_0 a_1^i \to \sum_{i=0}^{\infty} a_0 a_1^i = \frac{a_0}{1-a_1}$

Solution?

- Have $y_t = \sum_{i=0}^{t-1} a_0 a_1^i + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_1^t y_0$
- Want limit as push time 0 further back
- Use t_0 instead of 0 for initial time period which implies $y_t = \sum_{i=0}^{t-1-t_0} a_0 a_1^i + \sum_{i=0}^{t-1-t_0} a_1^i \varepsilon_{t-i} + a_1^{t-t_0} y_{t_0}$
- Let $h = t - t_0$ which implies $y_t = \sum_{i=0}^{h-1} a_0 a_1^i + \sum_{i=0}^{h-1} a_1^i \varepsilon_{t-i} + a_1^h y_{t-h}$
- $\lim_{h\to\infty} \sum_{i=0}^{h-1} a_0 a_1^i \to \sum_{i=0}^{\infty} a_0 a_1^i = \frac{a_0}{1-a_1}$
- $\lim_{h\to\infty} \sum_{i=0}^{h-1} a_1^i \varepsilon_{t-i} \to \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$ if ε_{t-1} is well behaved

Solution?

- Have $y_t = \sum_{i=0}^{t-1} a_0 a_1^i + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_1^t y_0$
- Want limit as push time 0 further back
- Use t_0 instead of 0 for initial time period which implies $y_t = \sum_{i=0}^{t-1-t_0} a_0 a_1^i + \sum_{i=0}^{t-1-t_0} a_1^i \varepsilon_{t-i} + a_1^{t-t_0} y_{t_0}$
- Let $h = t - t_0$ which implies $y_t = \sum_{i=0}^{h-1} a_0 a_1^i + \sum_{i=0}^{h-1} a_1^i \varepsilon_{t-i} + a_1^h y_{t-h}$
- $\lim_{h\to\infty} \sum_{i=0}^{h-1} a_0 a_1^i \to \sum_{i=0}^{\infty} a_0 a_1^i = \frac{a_0}{1-a_1}$
- $\lim_{h\to\infty} \sum_{i=0}^{h-1} a_1^i \varepsilon_{t-i} \to \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$ if ε_{t-1} is well behaved
- $\lim_{h\to\infty} a_1^h = 0$ and assume $\lim_{h\to\infty} \varepsilon_{t-h}$ is bounded so that $\lim_{h\to\infty} a_1^h \varepsilon_{t-h} = 0$

Solution?

- Have $y_t = \sum_{i=0}^{t-1} a_0 a_1^i + \sum_{i=0}^{t-1} a_1^i \varepsilon_{t-i} + a_1^t y_0$
- Want limit as push time 0 further back
- Use t_0 instead of 0 for initial time period which implies $y_t = \sum_{i=0}^{t-1-t_0} a_0 a_1^i + \sum_{i=0}^{t-1-t_0} a_1^i \varepsilon_{t-i} + a_1^{t-t_0} y_{t_0}$
- Let $h = t - t_0$ which implies $y_t = \sum_{i=0}^{h-1} a_0 a_1^i + \sum_{i=0}^{h-1} a_1^i \varepsilon_{t-i} + a_1^h y_{t-h}$
- $\lim_{h\to\infty} \sum_{i=0}^{h-1} a_0 a_1^i \to \sum_{i=0}^{\infty} a_0 a_1^i = \frac{a_0}{1-a_1}$
- $\lim_{h\to\infty} \sum_{i=0}^{h-1} a_1^i \varepsilon_{t-i} \to \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$ if ε_{t-1} is well behaved
- $\lim_{h\to\infty} a_1^h = 0$ and assume $\lim_{h\to\infty} \varepsilon_{t-h}$ is bounded so that $\lim_{h\to\infty} a_1^h \varepsilon_{t-h} = 0$
- Yielding $y_t = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_t$ as the solution

General method of solution

- Similar to way differential equations are solved

1. Solve the homogenous equation (no constant term)
2. Solve for a particular solution (complete equation)
3. The general solution is the sum of the particular solution and a linear combination of all homogeneous solutions
4. There will be an arbitrary constant which can be eliminated by imposing the initial condition on the general solution or by driving the solution back infinitely far

Rules for solving linear difference equations with constant coefficients

- Linear difference equation – nth order

$$y_t = a_0 + \sum_{i=1}^{n} a_i y_{t-i} \tag{3}$$

- Homogeneous form

$$y_t = \sum_{i=1}^{n} a_i y_{t-i} \tag{4}$$

1. If the homogenous equation (4) has n independent solutions $y = y_j(t)$, then $y = \sum_j y_j(t)$ also is a solution.
2. If the homogeneous equation (4) has the solution $y = y_j(t)$, then $y = Ay_j(t)$ also is a solution.
3. If $y = \overline{y}(t)$ is any particular solution whatever of equation (3) and if $y = y(t; A_1, A_2, \ldots, A_n)$ is a solution of (4), then the general solution of (3) is $y = \overline{y}(t) + y(t; A_1, A_2, \ldots, A_n)$

Solve first-order difference equation

- $y_t = a_0 + a_1 y_{t-1}$
- A possible particular solution is a constant $\overline{y}$
  - If $\overline{y}$ is the solution, then $\overline{y} = a_0 + a_1\overline{y}$ and if $a_1 \neq 1$, $\overline{y} = \frac{a_0}{1-a_1}$
- Solve homogeneous difference equation (equation without constant) $y_t^h = a_1 y_{t-1}^h$ where the superscript h just indicates the homogeneous solution
  - Guess $y_t = A\lambda^t$ where A is an arbitrary constant and λ is a parameter to be determined
  - Substituting into the homogeneous equation, we find $A\lambda^t = a_1 A\lambda^{t-1}$ which holds if $\lambda = a_1$ for any A

Solve first-order difference equation

- $y_t = a_0 + a_1 y_{t-1}$
  - Particular solution of $y_t = a_0 + a_1 y_{t-1}$ is $\bar{y}$
  - Homogeneous: $y_t^h = A\lambda^t$ is a solution to the homogeneous equation if $\lambda = a_1$
- General solution is
$$y_t = Aa_1^t + \frac{a_0}{1 - a_1} \tag{5}$$
- Value of A? At $t = 0$, equation (5) implies first-order autoregression is
$$y_0 = Aa_1^0 + \frac{a_0}{1 - a_1} = A + \frac{a_0}{1 - a_1} \Rightarrow A = y_0 - \frac{a_0}{1 - a_1}$$
and
$$y_t = \left(y_0 - \frac{a_0}{1 - a_1}\right) a_1^t + \frac{a_0}{1 - a_1}$$

Solve first-order difference equation

- In the general solution

$$y_t = \left(y_0 - \frac{a_0}{1 - a_1} \right) a_1^t + \frac{a_0}{1 - a_1}$$

  - The term $\left(y_0 - \frac{a_0}{1-a_1} \right) a_1^t$ represents transient behavior which
    - is zero if $y_0 = \frac{a_0}{1-a_1}$
    - goes to zero as initial period gets farther back in the past if $|a_1| < 1$

Solve first-order difference equation

What if slope coefficient equals 1?

- $y_t = a_0 + y_{t-1}$
- Solve homogeneous equation $y_t^h = a_1 y_{t-1}^h$
  - Guess $y_t^h = A\lambda^t$
  - Verify that $y_t^h = A\lambda^t$ is a solution with $\lambda = a_1 = 1$
- $\overline{y} = \frac{a_0}{1-a_1}$ can't be a particular solution
- Try $\overline{y} = bt$, b a constant for trend growth (bar means "particular", not "constant")
  - $y_t = a_0 + y_{t-1} \Rightarrow bt = a_0 + b\,(t-1)\,,\ bt = a_0 + bt - b,\ 0 = a_0 - b$
- General solution is

$$y_t = A + a_0 t$$

- Value of A? $y_0 = A + b \cdot 0,\ A = y_0$ and

$$y_t = y_0 + a_0 t$$

Solve first-order difference equation

What if slope coefficient is greater than 1?

- $y_t = a_0 + y_{t-1}$
- Solve homogeneous equation $y_t^h = a_1 y_{t-1}^h$
  - Guess $y_t^h = A\lambda^t$
  - Verify that $y_t^h = A\lambda^t$ is a solution with $\lambda = a_1$
- $\overline{y} = \frac{a_0}{1-a_1}$ can be a solution
  - Verify that $\overline{y} = \frac{a_0}{1-a_1}$ can be a solution
- General solution is

$$y_t = Aa_1^t + \frac{a_0}{1-a_1}$$

- with the value of A set, it is

$$y_t = \left(y_0 - \frac{a_0}{1-a_1}\right) a_1^t + \frac{a_0}{1-a_1}$$

- Transient behavior does not go to zero
  - Becomes increasingly large as take initial period increasingly far back in the past

Illustrate method of undetermined coefficients

- Guess a solution, called challenge solution
- Substitute it into difference equation and see if an identity in the coefficients emerges
- We already did this for the non-transient part with the first-order difference equation

$$y_t = a_0 + a_1 y_{t-1} \tag{6}$$

  - Guess answer is

$$y_t = \overline{y} \tag{7}$$

- Substitute (7) into (6)
- $\overline{y} = a_0 + a_1 \overline{y}$
- $\overline{y} = \frac{a_0}{1-a_1}$

First-order autoregression with forcing variable I

- Equation to solve is

$$y_t = a_0 + a_1 y_{t-1} + \varepsilon_t, \varepsilon_t \sim N(0,1), \ |a_1| < 1 \tag{8}$$

- $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$, $\varepsilon_t \sim N(0,1)$
- Iterate a few times
  - $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$
  - $y_t = a_0 + a_1 \left(a_0 + a_1 y_{t-2} + \varepsilon_{t-1}\right) + \varepsilon_t =$ $a_0 + a_1 a_0 + a_1^2 y_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$
  - $y_t = a_0 + a_1 a_0 + a_1^2 \left(a_0 + a_1 y_{t-3} + \varepsilon_{t-2}\right) + a_1 \varepsilon_{t-1} + \varepsilon_t =$ $a_0 + a_1 a_0 + a_1^2 a_0 + a_1^3 y_{t-3} + a_1^2 \varepsilon_{t-2} + a_1 \varepsilon_{t-1} + \varepsilon_t$

First-order autoregression with forcing variable I

- Pattern
  - Constant term with $a_0 + a_1 a_0 + a_1^2 a_0 + ... + a_1^i a_0$
  - $a_1^i y_{t-i}$
  - Innovations with $\varepsilon_t + a_1 \varepsilon_{t-1} + a_1^2 \varepsilon_{t-2} + ... + a_1^i \varepsilon_{t-i}$
- Implications as lag goes to infinity
  - As i goes to infinity, term $a_1^i y_{t-i}$ goes to zero
  - As i goes to infinity, sum of constants goes to $\frac{a_0}{1-a_1}$
  - As i goes to infinity, sum of innovations goes to $\sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}$
- Guess solution

$$y_t = b_0^* + \sum_{i=0}^{\infty} b_i \varepsilon_{t-i} \tag{9}$$

- Verify by substituting $y_t = b_0^* + \sum_{i=0}^{\infty} b_i \varepsilon_{t-i}$ into $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$

First-order autoregression with forcing variable II

- Substitute $y_t = b_0^* + \sum_{i=0}^{\infty} b_i \varepsilon_{t-i}$ into $y_t = a_0 + a_1 y_{t-1} + \varepsilon_t$

$$b_0^* + \sum_{i=0}^{\infty} b_i \varepsilon_{t-i} = a_0 + a_1 \left(b_0^* + \sum_{i=1}^{\infty} b_{i-1} \varepsilon_{t-i} \right) + \varepsilon_t$$

- Equating constants and coefficients of innovations, we see

$$\begin{aligned} b_0^* &= a_0 + a_1 b_0^* \\ b_0^* &= \frac{a_0}{1 - a_1} \\ b_i &= a_1 b_{i-1}, \;\; i = 1, \ldots, \infty \end{aligned}$$

First-order autoregression with forcing variable III

- which implies

$$
\begin{aligned}
b_0^* &= \frac{a_0}{1-a_1} \\
b_0 &= 1 \\
b_1 &= a_1 b_0 = a_1 \\
b_2 &= a_1 b_1 = a_1^2 \\
b_i &= a_1 b_{i-1} = a_1^i
\end{aligned}
$$

- $b_0 = 1$ is a normalization
- Therefore

$$
y_t = \frac{a_0}{1-a_1} + \sum_{i=0}^{\infty} a_1^i \varepsilon_{t-i}
$$

In terms of lag operator for slope coefficient less than one I

- $|a_1| < 1$
- Equation is

$$
\begin{aligned}
y_t &= a_0 + a_1 y_{t-1} \\
(1 - a_1 \mathrm{L})\, y_t &= a_0
\end{aligned}
$$

- Solution with period 0 is

$$y_t = \left(y_0 - \frac{a_0}{1 - a_1} \right) a_1^t + \frac{a_0}{1 - a_1}$$

- If period 0 is infinitely far back in the past, transient behavior is unimportant because $\lim \left(y_0 - \frac{a_0}{1-a_1} \right) a_1^t = 0$, and solution is

$$y_t = \frac{a_0}{1 - a_1}$$

In terms of lag operator for slope coefficient less than one II

- Can solve for this in terms of the lag operator by writing

$$y_t = \frac{a_0}{(1 - a_1 \mathrm{L})}$$

  - noting that

$$y_t = \sum_{i=0}^{\infty} a_1^i \mathrm{L}^i a_0 = \frac{a_0}{(1 - a_1)}$$

- Might seem like we're shutting down variation over time but more generally have a forcing variable x_t and

$$(1 - a_1 \mathrm{L})\, y_t = a_0 + x_t$$

In terms of lag operator for slope coefficient less than one III

- Equation can be written

$$y_t = \frac{a_0}{(1 - a_1\,\mathrm{L})} + \frac{1}{(1 - a_1\,\mathrm{L})}x_t$$

$$y_t = \frac{a_0}{(1 - a_1)} + \sum_{i=0}^{\infty} a_1^i x_t$$

Root of equation characterizes behavior I

- Root of difference equation is important for determining behavior
- A first-order difference equation is a polynomial of order 1 in in the lag operator

$$(1 - a_1 \, \mathrm{L}) \, y_t = a_0$$

- z transform
- Solve for root of polynomial in lag operator

$$(1 - a_1 \lambda) = 0$$

- Root is

$$\lambda = a_1^{-1}$$

- $|\lambda| > 1$ implies the equation is stable in the sense of returning to its mean in response to transient variation

Root of equation greater or less than one?

- Important digression: Root less than one or greater than one for stability?
  - Will see both characterizations
  - A root greater than one is necessary for stability – commonly for time-series statisticians
  - A root less than one is necessary for stability – sometimes for econometricians
- They both can be correct

$$(1 - a_1\lambda) = 0$$
$$\left(\lambda^f - a_1\right) = 0$$

- $\lambda > 1$ and $\lambda^f < 1$ are the same thing: $\lambda^{-1} = \lambda^f$

Possible strategy for solving difference equations I

- Solve for roots in polynomial in lag operator or forward operator
- We are interested in solving for roots to characterize behavior of series, not learning to be difference-equation wizards
- Can rely on Fundamental theorem of algebra
    - Every polynomial of order p has exactly p roots

Second-order difference equation I

- Second-order difference equation is

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-1}$$

- Can be written

$$\begin{aligned} (y_t - a_1 y_{t-1} - a_2 y_{t-2}) &= a_0 \\ \left(1 - a_1 \,\mathrm{L} - a_2 \,\mathrm{L}^2\right) y_t &= a_0 \end{aligned}$$

- There are two roots λ_i of this quadratic equation such that

$$\left(1 - a_1 \lambda_i - a_2 \lambda_i^2\right) = 0$$

- Solve for λ_1 and λ_2
- If $|\lambda_1| > 1$ and $|\lambda_2| > 1$, then the difference equation is stable
- If $|\lambda_1| = 1$ and $|\lambda_2| > 1$, then the difference equation has one unit root

Second-order difference equation II

- If $|\lambda_1| = 1$ and $|\lambda_2| = 1$, then the difference equation has two unit roots
- If $|\lambda_i| < 1$ for either or both roots, then the difference equation is explosive

Second-order difference equation with inverse roots (equally correct) I

- Equation is

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-1}$$

- Can be written

$$\begin{aligned} \left(y_{t+2} - a_1 y_{t+1} - a_2 y_t\right) &= a_0 \\ \left(\mathsf{F}^2 - a_1 \mathsf{F} - a_2\right) y_t &= a_0 \end{aligned}$$

- Every polynomial can be factored to yield

$$\left(\mathsf{F} - \lambda_1^f\right)\left(\mathsf{F} - \lambda_2^f\right) y_t = a_0$$

- Solve for λ_1^f and λ_2^f
- If $|\lambda_1^f| < 1$ and $|\lambda_2^f| < 1$, then the difference equation is stable

Second-order difference equation with inverse roots (equally correct) II

- If $|\lambda_1^f| = 1$ and $|\lambda_2^f| < 1$, then the difference equation has one unit root
- If $|\lambda_1^f| = 1$ and $|\lambda_2^f| = 1$, then the difference equation has two unit roots
- If $|\lambda_i^f| > 1$, for either or both roots, then the difference equation is explosive

Solving second-order difference equation I

- We are interested in solving for roots to characterize behavior of time series, not to learn to be difference-equation wizards
- Solve for roots of polynomial in lag operator or forward operator
- Can rely on Fundamental theorem of algebra
    - Every polynomial of order p has exactly p roots

General method

- Equation is

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-1}$$

- Can rely on Fundamental theorem of algebra
  - Every polynomial of order p has exactly p roots
  - That means two roots here

Particular solution

- The particular solution is

$$\bar{y} = a_0 + a_1\bar{y} - a_2\bar{y}$$

  and

$$\bar{y} = \frac{a_0}{1 - a_1 + a_2}$$

  if $a_1 + a_2 \neq 1$
- If $a_1 + a_2 = 1$, there is at least one unit root
- Suppose there is no unit root

Homogeneous solution to second-order difference equation I

- Homogeneous solution – guess $y_t^h = A\alpha^t$
  - Note: Using α here because the book does – same as λ^f above

$$A\alpha^t = a_1 A\alpha^{t-1} + a_2 A\alpha^{t-2}$$
$$\alpha^2 - a_1\alpha - a_2 = 0$$

- This characteristic equation has two roots given by the quadratic formula

$$\text{roots of } a\alpha^2 + b\alpha + c = 0 \text{ given by } \alpha_1, \alpha_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\alpha_1, \alpha_2 = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

- The roots can be real or imaginary

Two real roots I

- Characteristic equation

$$\alpha_1, \alpha_1 = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

- If $a_1^2 + 4a_2 > 0$, there are two real roots
  - $a_1^2 + 4a_2$ is called the "discriminant"
  - Absolute value of roots less than one for convergent behavior

Discriminant is zero I

- Characteristic equation

$$\alpha_1, \alpha_1 = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

- If $a_1^2 + 4a_2 = 0$, it might seem there is only one root $a_1/2$ but there is another one
  - The other root is $t\left(a_1/2\right)$ – a trend term
  - Probability of getting this with data is basically zero and I will move on

Discriminant is negative I

- Characteristic equation

$$\alpha_1, \alpha_1 = \frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

- If $a_1^2 + 4a_2 < 0$, there are two complex conjugate roots

$$\alpha_1, \alpha_1 = \frac{a_1 \pm i\sqrt{-\left(a_1^2 + 4a_2\right)}}{2}$$

  - where $i = \sqrt{-1}$
  - Obviously must appear as a pair with

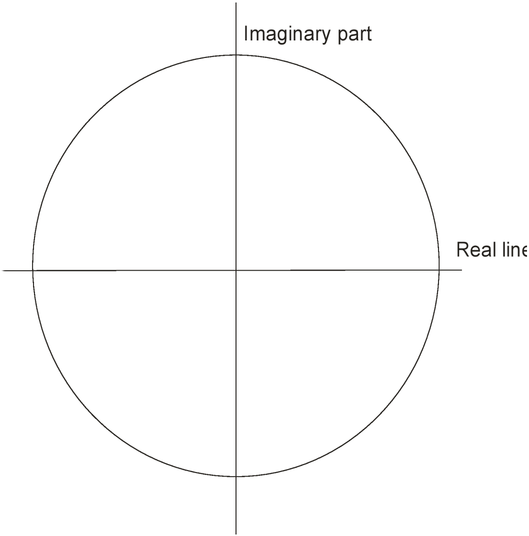
$$\alpha_1 = \frac{a_1 + i\sqrt{-\left(a_1^2 + 4a_2\right)}}{2}$$

$$\alpha_2 = \frac{a_1 - i\sqrt{-\left(a_1^2 + 4a_2\right)}}{2}$$

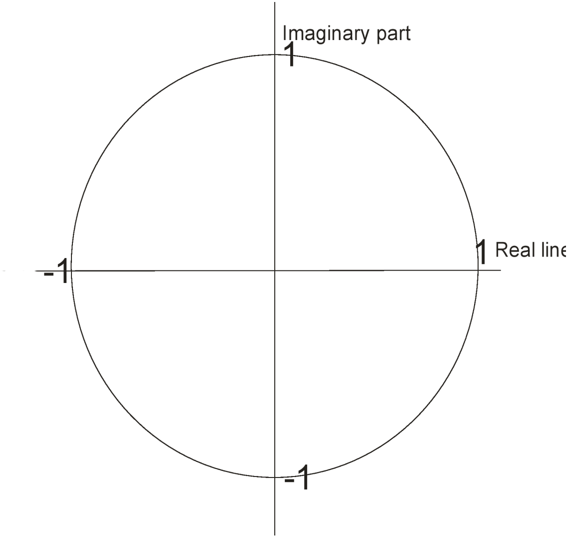
Discriminant is negative II

- Oscillatory behavior
- Can be stable or unstable
- How can we tell? The unit circle

Argand diagram with a circle super-imposed



Unit Circle



Why does unit circle work?

- Can characterize circle in terms of radius, here one, and radius for a circle from origin equals

$$r^2 = re^2 + im^2$$

  where r is the radius, re is the real part (or just x-axis) and im is the imaginary part (or just y-axis), all real numbers
- Can determine location of a complex root by squaring root
- Square of a complex number is the number $re + i \cdot im$ times its complex conjugate $re - i \cdot im$
- Square of $re + i \cdot im$ is $re^2 + im^2$
- Absolute value of $re^2 + im^2$ is $\left(re^2 + im^2\right)^{1/2}$ – distance from zero to the location of the number on the circle with this radius
- If roots are calculated as in textbook, (λ^f) are inside unit circle, stable
- If calculated as statisticians calculate roots (λ), roots outside the unit circle for stability

Complex roots and oscillations

- Complex roots can be associated with oscillations – recurrent cycles
- Can be seen through relationship of complex numbers and Argand diagram
- Simple way: spreadsheet

Summary

- Difference equations can be stable or unstable
  - Unstable means a trajectory that diverges
  - Stable means a trajectory that converges to a mean, to trend growth, oscillations, some long-run path that could be repeated forever
- The roots of a difference equation summarize the behavior
  - As typically calculated by economists, roots with an absolute value less than one are stable
- Second-order difference equations have two roots
  - The roots can be complex numbers
  - Even so, roots with an absolute value less than one are stability
- Unit roots mean the equation does not converge to a constant mean
- Unit roots do not mean the equation is "unstable" in the sense of diverging from trend behavior
  - $y_t = a_0 + y_{t-1}$ always has the value given by $y_t - y_{t-1} = a_0$
  - Stable in terms of growth, just not level