The Johansen Tests for Cointegration

Gerald P. Dwyer

April 2015

Time series can be cointegrated in various ways, with details such as trends assuming some importance because asymptotic distributions depend on the presence or lack of such terms. I will focus on the simple case of one unit root in each of the variables with no constant terms or other deterministic terms. These notes are a quick summary of some results without derivations.

Cointegration and Eigenvalues

The Johansen test can be seen as a multivariate generalization of the augmented Dickey-Fuller test. The generalization is the examination of linear combinations of variables for unit roots. The Johansen test and estimation strategy – maximum likelihood – makes it possible to estimate all cointegrating vectors when there are more than two variables.[1] If there are three variables each with unit roots, there are at most two cointegrating vectors. More generally, if there are n variables which all have unit roots, there are at most $n-1$ cointegrating vectors. The Johansen test provides estimates of all cointegrating vectors. Just as for the Dickey-Fuller test, the existence of unit roots implies that standard asymptotic distributions do not apply.

Slight digression for an assertion: If there are n variables and there are n cointegrating vectors, then the variables do not have unit roots. Why? Because the cointegrating vectors

[1] Results generally go through for quasi-maximum likelihood estimation.

can be written as scalar multiples of each of the variables alone, which implies that the variables do not have unit roots.

The vector autoregression (VAR) in levels with the constant suppressed is

$$\mathbf{x}_t = \sum_{i=1}^{k} \mathbf{A}_i \mathbf{x}_{t-i} + \mathbf{u}_t \tag{1}$$

For $k > 1$, this VAR in the levels always can be written

$$\Delta \mathbf{x}_t = \mathbf{\Pi} \mathbf{x}_{t-1} + \sum_{i=1}^{k-1} \mathbf{\Pi}_i \Delta \mathbf{x}_{t-i} + \mathbf{u}_t \tag{2}$$

For the simpler case $k = 1$, it is simply $\Delta \mathbf{x}_t = \mathbf{\Pi} \mathbf{x}_{t-1} + \mathbf{u}_t$

The matrix $\mathbf{\Pi}$ can be written in terms of the vector or matrix of adjustment parameters $\boldsymbol{\alpha}$ and the vector or matrix of cointegrating vectors $\boldsymbol{\beta}$ as

$$\mathbf{\Pi} = \boldsymbol{\alpha} \boldsymbol{\beta}' \tag{3}$$

For example, if the number of variables, n, is two and there is one cointegrating vector, then the vector $\boldsymbol{\beta}$ is $2 \,\mathrm{x}\, 1$ and the vector $\boldsymbol{\alpha}$ is $2 \,\mathrm{x}\, 1$. The two coefficients in the cointegrating vector $\boldsymbol{\beta}'$ multiply the variables to deliver the linear combination of variables that does not have a unit root, that is $\boldsymbol{\beta}' \mathbf{x}_{t-1}$. The two coefficients in $\boldsymbol{\alpha}$ are the two adjustment coefficients, one for each of the two equations, which multiply the cointegrating relationship $\boldsymbol{\beta}' \mathbf{x}_{t-1}$ to deliver the response of the variables in the two equations to deviations of the cointegrating relationship from zero.

If the matrix $\mathbf{\Pi}$ equals a matrix of zeroes, that is, $\mathbf{\Pi} = \mathbf{0}$ then the variables are not cointegrated and the relationship reduces to the vector autoregression in the first differences

$$\Delta \mathbf{x}_t = \sum_{i=1}^{k-1} \mathbf{\Pi}_i \Delta \mathbf{x}_{t-i} + \mathbf{u}_t \tag{4}$$

How can one test whether $\mathbf{\Pi} = \mathbf{0}$? One way is to test whether the rank of $\mathbf{\Pi}$ is zero, that is whether

$$\text{rank}\,(\mathbf{\Pi}) = 0 \tag{5}$$

If the variables are cointegrated, then $\text{rank}\,(\mathbf{\Pi}) \neq 0$ and in fact $\text{rank}\,(\mathbf{\Pi}) =$ the number of cointegrating vectors. The number of cointegrating vectors is less than or equal to the number of variables n and strictly less than n if the variables have unit roots.

If the rank of $\mathbf{\Pi}$ is less than n, then its determinant is zero. Eigenvalues are useful for solving this problem because the determinant of a square matrix equals the product of the eigenvalues. If the rank of the matrix is less than the number of rows and columns in the matrix, then one or more eigenvalues is zero and the determinant is zero.

What are eigenvalues? The set of eigenvalues for the $n \,\text{x}\, n$ matrix $\boldsymbol{A}$ are given by the n solutions to the polynomial equation

$$\det\,(\boldsymbol{A} - \lambda \mathbf{I}_n) = 0 \tag{6}$$

where $\mathbf{I}_n$ is an nth order identity matrix and det(.) denotes the determinant of the matrix $\boldsymbol{A} - \lambda \mathbf{I}_n$. Direct computation shows that equation (6) is an nth order polynomial, which has n not necessarily distinct roots.

The Johansen tests are based on eigenvalues of transformations of the data and represent linear combinations of the data that have maximum correlation (canonical correlations). To repeat, the eigenvalues used in Johansen's test are *not* eigenvalues of the matrix $\mathbf{\Pi}$ directly, although the eigenvalues in the test also can be used to determine the rank of $\mathbf{\Pi}$ and have tractable distributions. The eigenvalues are guaranteed to be non-negative and real. It would take us far afield to go into this and would serve little purpose in the end (other than torturing most of the people in class, which does not seems particularly desirable).[2]

Suppose that eigenvalues for the Johansen test have been computed.

[2]f you wish to pursue this topic, I suggest you study chapters 7 and 8 in *New Introduction to Multiple Time Series* by Helmut Lütkepohl.

Order the n eigenvalues by size so $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and recall that $\lambda_i \geq 0$ for all i. If $\lambda_1 = 0$, then the rank of $\mathbf{\Pi}$ is zero and there are no cointegrating vectors. If $\lambda_1 \neq 0$, then the rank of $\mathbf{\Pi}$ is greater than or equal to one and there is at least one cointegrating vector. If $\lambda_1 = 0$, stop with a conclusion of no cointegrating vectors.[3]

If $\lambda_1 \neq 0$, then continue testing by moving on to $\lambda_2 \leq \lambda_1$. If $\lambda_2 = 0$, then the rank of $\mathbf{\Pi}$ is one and there is one cointegrating vector. If $\lambda_2 \neq 0$, then the rank of $\mathbf{\Pi}$ is at least two and there are two or more cointegrating vectors.

And so on ....

If $\lambda_{n-1} \neq 0$, then test whether $\lambda_n = 0$. If $\lambda_n = 0$, then there are $n-1$ cointegrating vectors. If $\lambda_n \neq 0$, the variables do not have unit roots.

In an application with two variables, the maximum number of cointegrating vectors is two. Two cointegrating vectors would indicate that the variables do not have unit roots. The eigenvalues are λ_1 and λ_2 with $\lambda_1 > \lambda_2$. If $\lambda_1 = 0$, then there are no cointegrating vectors. If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, there is one cointegrating vector. If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, the variables do not have unit roots.

The Johansen Tests

The Johansen tests are called the maximum eigenvalue test and the trace test.

Let r be the rank of $\mathbf{\Pi}$. As the discussion above indicated, this is the same as the number of cointegrating vectors.

The Johansen tests are likelihood-ratio tests. There are two tests: 1. the maximum eigenvalue test, and 2. the trace test.

For both test statistics, the initial Johansen test is a test of the null hypothesis of no cointegration against the alternative of cointegration. The tests differ in terms of the alternative hypothesis

[3] If $\lambda_1 = 0$ and $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, then $\lambda_1 = 0 = \lambda_2 = ... = \lambda_n$

Maximum Eigenvalue Test

The maximum eigenvalue test examines whether the largest eigenvalue is zero relative to the alternative that the next largest eigenvalue is zero. The first test is a test whether the rank of the matrix $\mathbf{\Pi}$ is zero. The null hypothesis is that rank $(\mathbf{\Pi}) = 0$ and the alternative hypothesis is that rank $(\mathbf{\Pi}) = 1$. For further tests, the null hypothesis is that rank $(\mathbf{\Pi}) = 1, 2...$ and the alternative hypothesis is that rank $(\mathbf{\Pi}) = 2, 3, ...$

In more detail, the first test is the test of rank $(\mathbf{\Pi}) = 0$ and the alternative hypothesis is that rank $(\mathbf{\Pi}) = 1$. This is a test using the largest eigenvalue. If the rank of the matrix is zero, the largest eigenvalue is zero, there is no cointegration and tests are done. If the largest eigenvalue λ_1 is nonzero, the rank of the matrix is at least one and there might be more cointegrating vectors. Now test whether the second largest eigenvalue λ_2 is zero. If this eigenvalue is zero, the tests are done and there is exactly one cointegrating vector. If the second largest eigenvalue $\lambda_2 \neq 0$ and there are more than two variables, there might be more cointegrating vectors. Now test whether the third largest eigenvalue λ_3 is zero. And so on until the null hypothesis of an eigenvalue equal to zero cannot be rejected.

The test of the maximum (remaining) eigenvalue is a likelihood ratio test. The test statistic is

$$LR(r_0, r_0 + 1) = -T \ln (1 - \lambda_{r_0+1}) \tag{7}$$

where $LR(r_0, r_0 + 1)$ is the likelihood ratio test statistic for testing whether rank $(\mathbf{\Pi}) = r_0$ versus the alternative hypothesis that rank $(\mathbf{\Pi}) = r_0 + 1$. For example, the hypothesis that rank $(\mathbf{\Pi}) = 0$ versus the alternative that rank $(\mathbf{\Pi}) = 1$ is tested by the likelihood ratio test statistic $LR(0, 1) = -T \ln (1 - \lambda_1)$.

This likelihood ratio statistic does not have the usual asymptotic χ^2 distribution. This is similar to the situation for the Dickey-Fuller test: the unit roots in the data generate nonstandard asymptotic distributions.[4]

[4] There is a trivial case in which the distributions are standard (Enders, Appendix 6.2).

Trace Test

The trace test is a test whether the rank of the matrix $\mathbf{\Pi}$ is r_0. The null hypothesis is that $\text{rank}(\mathbf{\Pi}) = r_0$. The alternative hypothesis is that $r_0 < \text{rank}(\mathbf{\Pi}) \leq n$, where n is the maximum number of possible cointegrating vectors. For the succeeding test if this null hypothesis is rejected, the next null hypothesis is that $\text{rank}(\mathbf{\Pi}) = r_0 + 1$ and the alternative hypothesis is that $r_0 + 1 < \text{rank}(\mathbf{\Pi}) \leq n$.

Testing proceeds as for the maximum eigenvalue test.[5]

The likelihood ratio test statistic is

$$LR(r_0, n) = -T \sum_{i=r_0+1}^{n} \ln(1 - \lambda_i) \tag{8}$$

where $LR(r_0, n)$ is the likelihood ratio statistic for testing whether $\text{rank}(\mathbf{\Pi}) = r$ versus the alternative hypothesis that $\text{rank}(\mathbf{\Pi}) \leq n$. For example, the hypothesis that $\text{rank}(\mathbf{\Pi}) = 0$ versus the alternative that $\text{rank}(\mathbf{\Pi}) \leq n$ is tested by the likelihood ratio test statistic $LR(0, n) = -T\sum_{i=1}^{n} \ln(1 - \lambda_i)$.

Why is the trace test called the "trace test"? It is called the trace test because the test statistic's asymptotic distribution is the trace of a matrix based on functions of Brownian motion or standard Wiener processes (Johansen *Econometrica* 1995, p. 1555). That doesn't help much; most texts don't bother to explain the name at all because much more detail is necessary to say anything more informative.[6] Maybe a negative will be informative: The test is not based on the trace of $\mathbf{\Pi}$.

[5] A helpful thought to avoid some confusion is to keep in mind that the null hypothesis starts off from "not cointegrated".

[6] It is not hard to back out the actual matrix of eigenvalues which has the test statistic in equation (8) as the logarithm of its trace. It's also not illuminating.

Summary

This has been a quick run through the basics of the Johansen tests for cointegration.[7] There are many moving parts and the significance of them is not necessarily obvious from reading. Most people understand more of this after running a couple of tests for cointegration and puzzling over the results. That experience makes the whole thing much more concrete.

[7] The lecture goes through more details. Helmut Lütkepohl goes through the tests meticulously in chapters 7 and 8 in *New Introduction to Multiple Time Series*.